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MADRAS

ON THE MAXIMUM FUNCTION OF A MEROMORPHIC FUNCTION

By S. M. SHAH and S. K. SINGH

1. Let $f(z) = f_1(z)/P(z)$ be a meromorphic function with poles $b_1, b_2, \ldots$ and zeros $a_1, a_2, \ldots$. We suppose that $P(z)$ is a canonical product formed with $(b_n)_1^\infty$ as zeros. Let $\{C_i\}$ be the sequence of circles

$$|z - b_i| = \frac{1}{|b_i|^h} \quad (h > \rho_1,\ i = 1, 2, 3, \ldots),$$

ρ_1 being the exponent of convergence of the sequence (b_n). In the region D exterior to the circles C_i and interior to and on the circle $|z| = r$ define [1] the maximum function of $f(z)$ by

$$S(r) = \frac{M_1(r)}{m_2(r)},$$

where

$$M_1(r) = \max_{|z|=r} |f_1(z)| \text{ and } m_2(r) = \min_{|z|=r} |P(z)|,\ z \in D.$$

A number of results on the maximum function $S(r)$ have been proved by S. K. Bose (see [1] and his paper referred to therein). Unfortunately his Theorems 1, 2, 3 of [1] are false. We give here counter-examples.

THEOREM 1. *Given ρ, $0 < \rho < \infty$, there exists a meromorphic function $f(z) = f_1(z)/P(z)$ of order ρ such that*

$$\liminf_{r\to\infty} \frac{\log S(r)}{r^\rho} \neq \liminf_{r\to\infty} \frac{\log M_1(r)}{r^\rho}. \tag{1}$$

THEOREM 2. *Given ρ, $0 < \rho < \infty$, there exists a meromorphic function $f(z) = f_1(z)/P(z)$ of order ρ such that the entire function $f_1(z)$ is of order less than ρ and*

$$\liminf_{r\to\infty} \frac{\log S(r)}{r^\rho} = -\infty.$$

THEOREM 3. *Given $\rho > 0$ there exists a meromorphic function $f(z) = f_1(z)/P(z)$ of order ρ such that the entire function $f_1(z)$ is also of order ρ and*

$$\limsup_{r\to\infty} \frac{\log M_1(r) - \log m_2(r)}{n(r, 0)} = \infty.$$

2. LEMMA. *Let* $0 < \rho < \infty$,

$$R_n = \left(2^{2n} + \frac{1}{4}\right)^{1/\rho}, \quad \mu_n = (1 + [\rho])\, 2^{2n}, \; R = (2^{2k+1})^{1/\rho},$$

and let

$$P(z) = \prod_1^\infty \left\{ 1 - \left(\frac{z}{R_n}\right)^{\mu_n} \right\}^n .$$

Then $P(z)$ is a canonical product of order ρ and genus $[\rho]$; and on $|z| = R\ (> R_0)$

$$|P(z)| > A_1 M(R, P), \tag{2}$$

$$\log |P(z)| > A_2 (\log R) R^\rho. \tag{3}$$

For (see also Theorem 1 of [5]) the zeros of $P(z)$ are given by

$$\left(\frac{z}{R_n}\right)^{\mu_n} = e^{2s\pi i},$$

that is,

$$Z = R_n\, e^{2s\pi i/\mu_n} \; (s = 0, 1, 2, \dots, \mu_n - 1;\; n = 1, 2, \dots)$$

each with multiplicity n.

Further $n(R_n) = \sum_1^n m\mu_m > n\mu_n = (1 + [\rho])\, n2^{2n}$.

Let A denote a positive constant not necessarily the same at each occurrence, then

$$\Sigma n\mu_n < n\, \{2^2 + 2^{2.2} + \dots + 2^{2.n}\}\, (1 + [\rho]) < An2^{2n}.$$

Hence

$$\frac{\log n(R_n)}{\log R_n} \sim \frac{2n \log 2 + O(\log n)}{(1/\rho)\, 2n \log 2} \sim \rho,$$

so the order of $P(z)$ is ρ.

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Further

$$\sum \frac{n\mu_n}{R_n^{[\rho]}} = A \sum \frac{n\, 2^{2n}}{(2^{2n}+\frac{1}{4})^{[\rho]/\rho}}$$

is divergent and

$$\sum \frac{n\mu_n}{R_n^{1+[\rho]}} = A \sum \frac{n 2^{2n}}{(2^{2n}+\frac{1}{4})^{(1+[\rho])/\rho}}$$

is convergent. Hence the genus is $[\rho]$.

Now for $|z| = R = (2^{2k+1})^{1/\rho}$, where k is a large integer, we have

$$P(z) = \prod_{n=1}^{k} \left\{1 - \left(\frac{z}{R_n}\right)^{\mu_n}\right\}^n \prod_{n=k+1}^{\infty} \left\{1 - \left(\frac{z}{R_n}\right)^{\mu_n}\right\}^n = P_1\, P_2,$$

$$|P_1| \leqslant \prod_{n=1}^{k} \left\{1 + \left(\frac{R}{R_n}\right)^{\mu_n}\right\}^n = \prod_{n=1}^{k} \left\{\left(\frac{R}{R_n}\right)^{n\mu_n}\right\} \prod_{1}^{k} \left\{1 + \left(\frac{R_n}{R}\right)^{\mu_n}\right\}^n$$
$$= P_{11}\, P_{12}.$$

Now

$$P_{12} = \prod_{n=1}^{k} \left\{1 + \left(\frac{R_n}{R}\right)^{\mu_n}\right\}^n,$$

and

$$\frac{R_k}{R} = \frac{(2^{2k}+\frac{1}{4})^{1/\rho}}{(2^{2k+1})^{1/\rho}} \sim \frac{1}{2^{1/\rho}} < C < 1,$$

so

$$P_{12} \leqslant \prod_{1}^{k} \left\{1 + C^{\mu_n}\right\}^n$$

and $\sum nc^{\mu_n}$ is convergent; hence $P_{12} \leqslant A$.

Also

$$|P_2| \leqslant \prod_{n=k+1}^{\infty} \left\{1 + \left(\frac{R}{R_n}\right)^{\mu_n}\right\}^n,$$

and as

$$\frac{R}{R_{k+1}} = \frac{(2^{2k+1})^{1/\rho}}{(2^{2k+2}+\frac{1}{4})^{1/\rho}} < C_1 < 1,$$

so

$$|P_2| \leqslant A;$$

and hence

$$M(R, P) < A \prod_{n=1}^{k} \left(\frac{R}{R_n}\right)^{n\mu_n}$$

Further

$$|P_1| = \prod_1^k \left|1 - \left(\frac{z}{R_n}\right)^{\mu_n}\right|^n \geqslant \prod_1^k \left\{\left(\frac{R}{R_n}\right)^{\mu_n} - 1\right\}^n$$
$$= \prod_1^k \left(\frac{R}{R_n}\right)^{n\mu_n} \prod_1^k \left\{1 - \left(\frac{R_n}{R}\right)^{\mu_n}\right\}^n$$
$$= P_{11}\, P_{14};$$
$$\frac{R_n}{R} = \left(\frac{2^{2n} + \frac{1}{4}}{2^{2k+1}}\right)^{1/\rho} \sim \left(\frac{1}{2^{2k+1-2n}}\right)^{1/\rho}$$

Hence

$$\left(\frac{R_n}{R}\right)^{\mu_n} < C_2 < 1.$$

Hence

$$P_{14} > A > 0;$$

also

$$|P_2| > \prod_{n=k+1}^{\infty} \left\{1 - \left(\frac{R}{R_n}\right)^{\mu_n}\right\}^n > A > 0,$$

and so

$$m(R, P) > A \prod_1^k \left(\frac{R}{R_n}\right)^{n\mu_n} > A_1\, M(R, P)$$

Let

$$x = \prod_{n=1}^{k} \left(\frac{R}{R_n}\right)^{n\mu_n}.$$

Then

$$\log x = \sum_{n=1}^{k} n\mu_n \log\left(\frac{R}{R_n}\right)$$
$$\doteq \sum_1^k (1 + [\rho])\, n\, 2^{2n} \log\left(\frac{2^{2k+1}}{2^{2n} + \frac{1}{4}}\right)^{1/\rho},$$

so

$$\log x > A\, k\, 2^{2k} > A_3 (\log R) R^{\rho}.$$

Hence $\log |P(z)| > A_2 (\log R) R^\rho$ on $|z| = R$.

3. **Proof of Theorem 1.** First, suppose ρ is a non-integer. Let

$$f(z) = \frac{f_1(z)}{P(z)} = \frac{\prod_{\nu=2}^{\infty}\left\{\left(1-\frac{z}{a_\nu}\right)\exp\left(\frac{z}{a_\nu}+\dots+\frac{1}{q}\left(\frac{z}{a_\nu}\right)^q\right)\right\}}{\prod_{n=1}^{\infty}\left\{1-\left(\frac{z}{R_n}\right)^{\mu_n}\right\}^n},$$

where $q = [\rho]$, $a_\nu = -(\nu)^{1/\rho}$. Then $f(z)$ is a meromorphic function of order ρ and $f_1(z)$ and $P(z)$ both are of order ρ. Further, the circle $|z| = R = (2^{2k+1})^{1/\rho}$ lies in that part of the z-plane from which the circles

$$\left|z - R_n \exp\left(\frac{2p\pi i}{\mu_n}\right)\right| = \frac{1}{R_n^h};\ h > \rho\ (p = 0, 1, 2, \dots, \mu_n - 1;$$
$$n = 1, 2, \dots)$$

have been removed, and [2; p. 18]

$$\frac{\log S(R)}{R^\rho} = \frac{\log M_1(R) - \log m_2(R)}{R^\rho} < \frac{AR^\rho - A_2 (\log R) R^\rho}{R^\rho}$$
$$\to -\infty, \text{ as } R \to \infty.$$

Hence

$$\liminf_{r\to\infty} \frac{\log S(r)}{r^\rho} = -\infty.$$

Also [2; p. 18]

$$\liminf_{r\to\infty} \frac{\log M_1(r)}{r^\rho} = \limsup_{r\to\infty} \frac{\log M_1(r)}{r^\rho} = C,$$

where $0 < C < \infty$. Hence (1) is proved when ρ is non-integer. If ρ be an integer, we take

$$f(z) = \frac{e^{z^\rho}}{P(z)}.$$

Then for $|z| = R = (2^{2k+1})^{1/\rho}$

$$\frac{\log S(R)}{R^\rho} = \frac{\log M_1(R) - \log m_2(R)}{R^\rho}$$
$$< \frac{AR^\rho - A_2 (\log R) R^\rho}{R^\rho} \to -\infty, \text{ as } R \to \infty,$$

and

$$\frac{\log M_1(r)}{r^\rho} \to 1, \quad \text{as } r \to \infty;$$

hence, for the function $f(z)$,

$$\liminf_{r\to\infty} \frac{\log S(r)}{r^\rho} \neq \liminf_{r\to\infty} \frac{\log M_1(r)}{r^\rho}$$

and (1) is proved.

Similarly we can construct functions for which

$$\limsup_{r\to\infty} \frac{\log S(r)}{r^\rho} \neq \limsup_{r\to\infty} \frac{\log M_1(r)}{r^\rho}.$$

4. **Proof of Theorem 2.** Let

$$f(z) = \frac{f_1(z)}{P(z)} = \frac{\prod_{\nu=2}^{\infty}\left\{\left(1-\frac{z}{a_\nu}\right)\exp\left(\frac{z}{a_\nu}+\dots+\frac{1}{q}\left(\frac{z}{a_\nu}\right)^q\right)\right\}}{\prod_{1}^{\infty}\left\{1-\left(\frac{z}{R_n}\right)^{\mu_n}\right\}^n},$$

where $a_\nu = -(\nu)^{1/\rho_1}$, ρ_1 non-integer and $< \rho$, ρ integer or non-integer and $q = [\rho_1]$; then $f(z)$ is a meromorphic function of order ρ and the circle $|z| = R = (2^{2k+1})^{1/\rho}$ lies in that part of the z-plane from which the circles

$$\left|z - R_n \exp\left(\frac{2p\pi i}{\mu_n}\right)\right| = \frac{1}{R_n^h}, \; h > \rho,$$

$$(p = 0, 1, 2, \dots, \mu_n - 1; n = 1, 2, \dots),$$

have been removed; and

$$\frac{\log S(R)}{R^\rho} = \frac{\log M_1(R) - \log m_2(R)}{R^\rho}$$

$$< \frac{A\, R^{\rho_1} - A_2 R^\rho (\log R)}{R^\rho}$$

$$\to -\infty \text{ as } R \to \infty.$$

This proves Theorem 2.

5. **Proof of Theorem 3.** Let

$$f(z) = \frac{\prod_{1}^{\infty} \left\{ 1 + \frac{z^k}{\lambda_n} \right\}^{\xi_n}}{\prod_{1}^{\infty} \left\{ 1 - \frac{z}{((n+\frac{1}{2})^{n^n})^{1/k}} \right\}} = \frac{f_1(z)}{P(z)},$$

where

$$\lambda_n = n^{n^n},\ \epsilon_n = \frac{(4+\rho) \log \lambda_{n-1}}{\log \lambda_n}, \quad k = 1 + [\rho],$$

$$\xi_n = [\lambda_n^{\rho/k+\epsilon_n}],\ n = 1, 2, 3, \dots ;$$

here $f_1(z)$ is a canonical product of order ρ for

$$n(\lambda_n^{1/k}, f_1) = k(\xi_1 + \dots + \xi_n) \sim k\, \xi_n,$$

$$\frac{\log n(\lambda_n^{1/k}, f_1)}{\log (\lambda_n^{1/k})} \sim \frac{\log \xi_n}{(1/k) \log \lambda_n} \sim \frac{(\rho/k + \epsilon_n) \log \lambda_n}{(1/k) \log \lambda_n} \sim \rho.$$

Clearly $P(z)$ is an entire function of order zero.

Further $f_1(z)$ and $P(z)$ have no common zeros, because the zeros of $P(z)$ are all positive whereas the zeros of $f_1(z)$ are given by

$$z^k = -\lambda_n = \lambda_n e^{i\pi(1+2s)},$$

that is,

$$z = \lambda_n^{1/k} e^{i\pi(1+2s)/k},$$

and if some zeros of $f_1(z)$ are positive, then we must have

$$\frac{i\pi(1+2s)}{k} = 2m\pi i,$$

that is, $1 + 2s = 2mk$ which is impossible.

If D denotes that part of the z-plane from which the circles

$$\left| z - \left((n+\tfrac{1}{2})^{n^n} \right)^{1/k} \right| = \frac{1}{\left((n+\frac{1}{2})^{n^n} \right)^{h/k}},\ h > 0,\ n = 1, 2, 3, \dots$$

have been removed, then the circle $|z| = R = \frac{1}{2} \lambda_n^{1/k}$ lies in D and

$$\frac{\log M_1(R) - \log m_2(R)}{n(R, 0)} > \frac{\log M_1(R) - k \frac{(\log R)(\log \log R)}{\log \log \log R}}{n(R, 0)}.$$

Now

$$n(R, 0) \sim k\, \xi_{n-1} \sim k\, \lambda_{n-1},$$

and so

$$\frac{(\log R)\,(\log\log R)}{\log\log\log R} = o\,(n\,(R, 0)).$$

Also [3]

$$\log M_1(R) > \xi_n \log\left(1 + \frac{1}{2^k}\right),$$

and so

$$\limsup_{R\to\infty} \frac{\log M_1(R) - \log m_2(R)}{n(R, 0)} = \limsup_{R\to\infty} \frac{\log M_1(R)}{n(R, 0)} = \infty. \qquad (4)$$

This proves Theorem 3.

The relation (4) holds even when $n\,(R, 0)$ is multiplied by a function of R such as R^ρ (see [4]; [5, Theorem 2]).

We have given in full the proofs of (2) and (3) for the proof of Theorem 1 depends on (2) and (3).

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SUPERPOSITION OF ORTHOGONAL FLOWS

By PREM PRAKASH

Introduction. One of the applications of the theory of superposition of fluid flows is that we come across interesting examples of solutions of Navier-Stokes equations. The object of this paper is to investigate the type of two dimensional orthogonal flows superposable on each other. In course of investigations we have come across an interesting but known example of decay of vortices in a viscous fluid.

Two flows (u_1, v_1) and (u_2, v_2) are said to be superposable on each other provided the flow $(u_1 + u_2, v_1 + v_2)$ is also a valid flow, i.e. $u_1 + u_2$ and $v_1 + v_2$ are solution of Navier-Stokes equations for a viscous fluid motion.

The flows in each case belonging to a conservative system (u_1, v_1) and (u_2, v_2) will each satisfy [1]

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \nu \nabla^2\right)\zeta = 0. \tag{1}$$

The condition of superposability of two flows is [5]

$$u_1\frac{\partial \zeta_2}{\partial x} + u_2\frac{\partial \zeta_1}{\partial x} + v_1\frac{\partial \zeta_2}{\partial y} + v_2\frac{\partial \zeta_1}{\partial y} = 0. \tag{2}$$

If the two flows are orthogonal they have to satisfy in addition to (2) the condition

$$u_1 u_2 + v_1 v_2 = 0.$$

The general treatment of the problem is involved. If however $u_2 = 0$, then in the case of two separate flows v_1 must be zero and the two orthogonal superposable flows will be $(u_1, 0)$ and $(0, v_2)$.

We propose to investigate here what these flows can possibly be.

From the equation of continuity we get

$$u_1 = f(y), \text{ and } v_2 = \phi(x),$$

so that (2) reduces to

$$u_1 \frac{d^2 v_2}{dx^2} - v_2 \frac{d^2 u_1}{dy^2} = 0,$$

i.e.

$$\frac{\dfrac{d^2 u_1}{dy^2}}{u_1} = \frac{\dfrac{d^2 v_2}{dx^2}}{v_2} = -k, \text{say}, \tag{3}$$

where k is a positive constant or a function of time t. It can be easily seen that the use of a positive constant can make the solutions of (3) more fit for practical applications.

Solutions of (3) are

$$u_1 = A \cos(\sqrt{k}y + B), \tag{4}$$

$$v_2 = A_1 \cos(\sqrt{k}x + B_1), \tag{5}$$

where A, A_1, B and B_1 are either constants or functions of t. Substituting (4) in (1) we get

$$\left(\sqrt{k}\frac{dA}{dt} + \frac{A}{2\sqrt{k}}\frac{dk}{dt} + \nu A k^{3/2}\right)\sin(\sqrt{k}y + B) -$$

$$- A\sqrt{k}\left(\frac{y}{2\sqrt{k}}\frac{dk}{dt} + \frac{dB}{dt}\right)\cos(\sqrt{k}y + B) = 0.$$

Since this is an identity, coefficients of $\sin(\sqrt{k}y + B)$ and $\cos(\sqrt{k}y + B)$ must separately vanish. Hence k and B are absolute constants, and

$$\sqrt{k}\frac{dA}{dt} + \nu A k^{3/2} = 0,$$

so that,

$$A = C e^{-\nu kt}, \; C \text{ being an arbitrary constant.}$$

From (4) therefore

$$u_1 = C e^{-\nu kt} \cos(\sqrt{k}y + B), \tag{6}$$

and similarly,

$$v_2 = C_1 e^{-\nu kt} \cos(\sqrt{k}x + B_1), \tag{7}$$

C_1 and B_1 being arbitrary constants.

The boundaries for (6) and (7) are the straight parallel walls $y = \frac{2n+1}{2\sqrt{k}}\pi - \frac{B}{\sqrt{k}}$ and $x = \frac{2m+1}{2\sqrt{k}}\pi - \frac{B_1}{\sqrt{k}}$ respectively, n and m taking any integral values from $-\infty$ to $+\infty$. These velocities remain finite at large distances and decay with time, the rate depending on the kinematic viscosity. The vorticity is greatest at the walls, where the velocity is zero, and is steadily diffused towards the mid-stream.

The superposed flow. The superposed flow

$$\psi = \frac{e^{-\nu kt}}{\sqrt{k}}\left\{C_1 \sin(\sqrt{k}x + B_1) - C \sin(\sqrt{k}y + B)\right\}, \qquad (8)$$

is also a solution of the Navier-Stokes equations and hence represents a valid two-dimensional flow.

It is easily seen that (6), (7) and (8) are all self-superposable flows and so the vorticity will be constant following a stream line [2].

The vorticity of (8) is given by

$$\zeta = \sqrt{k}\, e^{-\nu kt}\left\{C \sin(\sqrt{k}y + B) - C_1 \sin(\sqrt{k}x + B_1)\right\}.$$

The vorticity is therefore a maximum or a minimum at the points

$$\left(\frac{2m+1}{2\sqrt{k}}\pi - \frac{B_1}{\sqrt{k}}, \frac{2n+1}{2\sqrt{k}}\pi - \frac{B}{\sqrt{k}}\right),$$ m and n being any integers.

These are the points where the velocity is evidently zero and we have vortices with vorticity $C - C_1$ or $C + C_1$. This therefore furnishes an example of decay of vortices in a viscous fluid like the one given by Taylor [7], his result being

$$\psi = e^{-2\pi^2\nu t/d^2}.\ A \cos \pi x/d \cos \pi y/d.$$

If in (8) we put $B_1 = \pi/2$, $B = 3\pi/2$, $C = C_1$, $\sqrt{(k/2)} = \pi/d$ and rotate the coordinate axes through $45°$, we get Taylor's result.

It is readily seen that no irrotational flow can be superimposed on (8). For if ψ be the stream function of such a flow we have from (2)

$$\psi = f[\sqrt{k}\, e^{-\nu kt}\{C \sin(\sqrt{k}y + B) - C_1 \sin(\sqrt{k}x + B_1)\}],$$

f being any function so that

$$\begin{aligned} \zeta &= \nabla^2 \psi \\ &= k^2 e^{-2\nu kt}\{C^2 \cos^2(\sqrt{k}y + B) + C_1^2 \cos^2(\sqrt{k}x + B_1)\} \times f'' - \\ &\qquad - k^{3/2} e^{-\nu kt}\{C \sin(\sqrt{k}y + B) - C_1 \sin(\sqrt{k}x + B_1)\}.f' \\ &= 0. \end{aligned}$$

This gives

$$\frac{f''}{f'} = \frac{C \sin(\sqrt{k}y + B) - C_1 \sin(\sqrt{k}x + B_1)}{C^2 \cos^2(\sqrt{k}y + B) + C_1^2 \cos^2(\sqrt{k}x + B_1)} \cdot \frac{1}{\sqrt{k}\, e^{-\nu kt}},$$

where dashes denote differentiation with respect to $\{C \sin(\sqrt{k}y + B) - C_1 \sin(\sqrt{k}x + B_1)\}$. Since the left hand side in this equation is a function of $\{C \sin(\sqrt{k}y + B) - C_1 \sin(\sqrt{k}x + B_1)\}$ and t, the right hand side must also be so. Therefore $C^2 \cos^2(\sqrt{k}y + B) + C_1^2 \cos^2(\sqrt{k}x + B_1)$ must be a function of $\{C \sin(\sqrt{k}y + B) - C_1 \sin(\sqrt{k}x + B_1)\}$. But the Jacobian of $C^2 \cos^2(\sqrt{k}y + B) + C_1^2 \cos^2(\sqrt{k}x + B_1)$ and $\{C \sin(\sqrt{k}y + B) - C_1 \sin(\sqrt{k}x + B_1)\}$ with respect to x and y vanishes only if either $C = 0$ or $C_1 = 0$. When C or C_1 is zero, (8) reduces to (7) or (6) and proceeding on the above lines it can be seen that the only irrotational flows superposable on (6) and (7) are the constant velocities along the flows themselves.

If we proceed with the parallel problem in polar coordinates by taking the two orthogonal flows $(u_r, 0)$ and $(0, v_\theta)$, we get from the corresponding equations of continuity that

$$u_r = f(\theta)/r, \; v_\theta = \phi(r).$$

Condition of superposability of u_r and v_θ gives

$$\frac{f''(\theta)}{f(\theta)} = \frac{r^2 \phi''(r) + r\phi'(r) - \phi(r)}{\phi(r)} = C = -k, \text{ say},$$

where C and k can be functions of time only or constants.

The two values of $f(\theta)$ are

$$f(\theta) = A \cos(\sqrt{k}\,\theta + B),\; f(\theta) = A_1 e^{\sqrt{c}\theta} + B_1 e^{-\sqrt{c}\theta},$$

where A, B, A_1 and B_1 are either constants or functions of time; so that we have

$$u_r = \frac{A \cos(\sqrt{k}\,\theta + B)}{r} \quad \text{or} \quad \frac{A_1 e^{\sqrt{c}\theta} + B_1 e^{-\sqrt{c}\theta}}{r}.$$

But it can easily be seen that with these values of u_r, the equations of motion are not exactly integrable and hence the solutions will not be of much interest. A flow superposable on $u_r = \dfrac{f(\theta)}{r}$ under non-conservative forces has been the subject of study by me in another paper which is under publication.

If however $f(\theta) = \text{constant} = a$, the flow u_r is irrotational and due to a source at the origin. It is known that the only flows superposable on it then are either irrotational [3] or those with constant vorticity [4]. In the latter case therefore $\phi = Ar$, A being a constant and represents a flow inside a circular cylinder rotating about the z-axis with angular velocity A [6].

The superposed flow. The superposed flow (u_r, v_θ) is given by

$$u_r = \frac{a}{r}, \quad v_\theta = Ar. \tag{9}$$

Since the velocity depends on r alone, the cylinder $r = r_1$ is a boundary provided it has the prescribed motion

$$u_{r_1} = \frac{a}{r_1}, \quad v_{\theta_1} = Ar_1,$$

i.e. it is rotating about the z-axis with constant angular velocity A and simultaneously undergoing radial elongation at the rate of $\dfrac{a}{r_1}$ per unit time. To avoid the infinite velocity at the origin, we can

again subject the flow to a similar boundary $r = r_2$, having the prescribed motion

$$u_{r_2} = \frac{a}{r_2}, \quad v_{\theta_2} = Ar_2,$$

provided the space between the coaxial cylinders remains constant during their motion and thus remains occupied by the liquid. For this we observe that

$$\frac{dr_1}{dt} = \frac{a}{r_1},$$

i.e.

$$r_1^2 = 2at + \text{constant}$$
$$= 2at + B^2, \tag{10}$$

when $r_1 = B$ for $t = 0$.

Similarly

$$r_2^2 = 2at + C^2, \tag{11}$$

where at $t = 0$, $r_2 = C$, so that

$$\pi(r_1^2 - r_2^2) = \pi(B^2 - C^2) = \text{constant},$$

at any time and for any values of r_1 and r_2.

The stream lines of the superposed flow are the spiral curves

$$r^2 = k\theta,$$

where k is a constant.

My thanks are due to Dr. Ram Ballabh for his guidance and to the Scientific Research Committee, U.P. for the monetary help given to me.

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Lucknow University

THE IDEAL STRAIGHT LINES ON THE CATENARY SURFACE AND ITS ADJOINED SURFACE

By M. PINL

As is well known all minimal surfaces are translation surfaces of isotropic curves [1]. The catenary surface and its adjoined surface are special minimal surfaces. According to a general theorem of S. Lie [2] their intersections with the ideal plane $x_0 = 0$ of the extended Euclidean space of coordinates x, y, z must be (ideal) straight lines. The adjoined surface of the catenary surface is the minimal screw surface. In the following, we shall prove that both surfaces have common ideal straight lines.

1. Catenary surface. The catenary surface is given by the equation

$$\operatorname{ch} \frac{z}{a} = \frac{\sqrt{(x^2 + y^2)}}{a}. \tag{1}$$

The homogeneous coordinates corresponding to the Cartesian coordinates x, y, z may be denoted by x_0, x_1, x_2, x_3. Then we get

$$x = \frac{x_1}{x_0},\ y = \frac{x_2}{x_0},\ z = \frac{x_3}{x_0}.$$

Hence the equation of the catenary surface can be written :

$$\left.\begin{aligned} \frac{\sqrt{(x_1^2 + x_2^2)}}{a\,x_0} &= \cos\left(\frac{i\,x_3}{a\,x_0}\right), \\ \operatorname{arc}\cos\left(\frac{\sqrt{(x_1^2 + x_2^2)}}{a\,x_0}\right) &= \frac{i\,x_3}{a\,x_0} = i \log\left(\frac{x_3}{a\,x_0} + i\sqrt{\left(1 - \frac{x_3^2}{a^2 x_0^2}\right)}\right), \end{aligned}\right\} \tag{2}$$

or

$$\begin{aligned} x_3 &= a\,x_0 \log \frac{x_3 + i\sqrt{(a^2 x_0^2 - x_3^2)}}{a\,x_0} \\ &= a\,x_0 \left[\log\left(x_3 + i\sqrt{(a^2 x_0^2 - x_3^2)}\right) - \log a\,x_0\right]. \end{aligned} \tag{3}$$

In order to get the ideal straight lines on the surface (2), we must discuss the limit process $x_0 \to 0$. By putting $x_3/x_0 = \xi + i\eta$ we get from (2)

$$\sqrt{(x_1^2 + x_2^2)} = \frac{a}{2} x_0 \left(e^{i\eta/a} e^{\xi/a} + e^{-i\eta/a} e^{-\xi/a}\right). \qquad (4)$$

Let $x_0 \to 0$, ξ being bounded. *Thus we obtain the pair of conjugate imaginary straight lines on the catenary surface* :

$$S_2: x_1 + i x_2 = 0, x_0 = 0, \ S_3: x_1 - i x_2 = 0, x_0 = 0.$$

Now we consider the equation (3) in the limit case $x_0 \to 0$. Since $x_0 \log x_0 \to 0$, if $x_0 \to 0$, we obtain from (3), $x_3 \to 0$, if $x_0 \to 0$, or *the real straight line*

$$S_1: x_3 = 0, x_0 = 0$$

is an ideal straight line on the catenary surface.

The points of intersections S_{12}, S_{13}, S_{23} of the ideal straight lines S_1, S_2, S_3 are given by the homogeneous coordinates :

$$S_{23} = (0, 0, 0, 1), S_{12} = (0, 1, i, 0), S_{13} = (0, 1, -i, 0).$$

These points lie on the absolute conic :

$$x_1^2 + x_2^2 + x_3^2 = 0, x_0 = 0, \qquad (5)$$

except of point S_{23}. Thus we get the result : *the ideal polar triangle* (S_1, S_2, S_3) *with respect to the absolute conic* (5) *lies on the catenary surface.*

2. Minimal screw surface. The minimal screw surface is given by the equation

$$z = \frac{x_3}{x_0} = a \operatorname{arc\,tg} \frac{y}{x} = a \operatorname{arc\,tg} \frac{x_2}{x_1} = \frac{a}{2} i \log \frac{1 - \frac{i x_2}{x_1}}{1 + \frac{i x_2}{x_1}}$$

$$= \frac{a}{2} i \log \frac{x_1 - i x_2}{x_1 + i x_2} = \frac{a}{2} i \log (x_1 - i x_2) - \frac{a}{2} i \log (x_1 + i x_2); \qquad (6)$$

or

$$x_3 = \frac{i}{2} a x_0 [\log (x_1 - i x_2) - \log (x_1 + i x_2)]. \qquad (7)$$

The limit process $x_0 \to 0$ when applied to (7) leads to $x_3 \to 0$. *The real ideal straight line*

$$S_1 : x_0 = 0, \; x_3 = 0,$$

is an ideal straight line on the minimal screw surface (6). Equation (6) can be modified as follows :

$$\sec^2\left(\frac{x_3}{a\,x_0}\right) = \cos^{-2}\left(\frac{x_3}{a\,x_0}\right) = \left[\frac{1}{2}\left(e^{i\,x_3/a\,x_0} + e^{-i\,x_3/a\,x_0}\right)\right]^{-2}$$

$$= 1 + \operatorname{tg}^2\left(\frac{x_3}{a\,x_0}\right) = \frac{x_1^2 + x_2^2}{x_1^2}$$

Putting $\frac{x_3}{a\,x_0} = \xi + i\eta$ we obtain

$$\sec\left(\frac{x_3}{a\,x_0}\right) = \left[\frac{1}{2}\left(e^{(i/a)(\xi+i\eta)} + e^{-(i/a)(\xi+i\eta)}\right)\right]^{-1}$$

$$= \left[\frac{1}{2}\left(e^{i\xi/a}\,e^{-\eta/a} + e^{i\xi/a}\,e^{\eta/a}\right)\right]^{-1} = \frac{\sqrt{(x_1^2 + x_2^2)}}{x_1}.$$

Let $\eta \to \infty$. Then we get (x_3 being bounded) :

$$\lim_{x\to 0} \sec\left(\frac{x_3}{a\,x_0}\right) = \lim_{\eta\to\infty} \frac{2}{e^{-\eta/a}\,e^{i\xi/a} + e^{\eta/a}\,e^{i\xi/a}} \to 0.$$

Hence

$$\frac{x_1^2 + x_2^2}{x_1^2} \to 0\,;\; (x_1 + i\,x_2)\,(x_1 - i\,x_2) \to 0.$$

The minimal screw surface (6) *and the catenary surface* (1) *have the common ideal straight lines*

$$S_1 : x_0 = 0, \; x_3 = 0\,; \; S_2 : x_1 + i\,x_2 = 0, \; x_0 = 0\,;$$
$$S_3 : x_1 - i\,x_2 = 0, \; x_0 = 0.$$

This result was to be expected, because the deformations which transform the catenary surface into the minimal screw surface and vice versa are affine transformations.

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Darjeeling

MATHEMATICAL NOTES

On triangular numbers in arithmetical progression

By Hansraj Gupta, *Punjab University College, Hoshiarpur*

1. M. N. Khatri has asked for a general solution of the problem of triangular numbers in arithmetical progression. I give the solution here in the hope that it would interest other readers also.

2. If $\Delta_a, \Delta_b, \Delta_c$ are in A.P., we must have

$$(b-a)\,(b+a+1) = (c-b)\,(c+b+1).$$

This is of the form

$$uv = xy.$$

Let

$$(u, x) = \lambda_1, \left(\frac{u}{\lambda_1}, y\right) = \lambda_2,$$

then

$$u = \lambda_1\,\lambda_2, x = \lambda_1\,\lambda_3, y = \lambda_2\,\lambda_4 ; v = \lambda_3\,\lambda_4.$$

where

$$(\lambda_2, \lambda_3) = 1.$$

Hence in our problem

$$b - a = \lambda_1\,\lambda_2, b + a + 1 = \lambda_3\,\lambda_4,$$

$$c - b = \lambda_1\,\lambda_3, c + b + 1 = \lambda_2\,\lambda_4,$$

$$\lambda_1\,\lambda_2 + \lambda_3\,\lambda_4 = 2b + 1 = \lambda_2\,\lambda_4 - \lambda_1\,\lambda_3,$$

or

$$\lambda_1\,(\lambda_2 + \lambda_3) = \lambda_4\,(\lambda_2 - \lambda_3).$$

If

$$(\lambda_1, \lambda_4) = \mu, \text{ and } (\lambda_2 + \lambda_3, \lambda_2 - \lambda_3) = \delta,$$

then

$$\lambda_1 = \mu(\lambda_2 - \lambda_3)/\delta,$$
$$\lambda_4 = \mu(\lambda_2 + \lambda_3)/\delta.$$

Hence

$$a = \tfrac{1}{2}\left\{\frac{\mu}{\delta}\left(\lambda_3^2 + 2\lambda_2\lambda_3 - \lambda_2^2\right) - 1\right\},$$

$$b = \tfrac{1}{2}\left\{\frac{\mu}{\delta}\left(\lambda_2^2 + \lambda_3^2\right) - 1\right\},$$

$$c = \tfrac{1}{2}\left\{\frac{\mu}{\delta}\left(\lambda_2^2 + 2\lambda_2\lambda_3 - \lambda_3^2\right) - 1\right\}.$$

Since

$$(\lambda_2, \lambda_3) = 1,$$

evidently a, b, c shall be integers if and only if μ is odd. Take

$$\lambda_2 = 4,\ \lambda_3 = 3,\ \mu = 5;$$

then

$$a = 42,\ b = 62,\ c = 77.$$

3. If Δ_a, Δ_b, Δ_c are in A.P., and Δ_x, Δ_y, Δ_z are in A.P., then Δ_{a+x}, Δ_{b+y}, Δ_{c+z} are in A.P., if and only if ax, by, cz are in A.P., for

$$(b-a+y-x)(b+a+y+x+1) = (c-b+z-y)(c+b+z+y+1),$$

if and only if

$$(b-a)(b+a+1)+(y-x)(y+x+1)+(b-a)(y+x)+(y-x)(b+a)$$
$$= (c-b)(c+b+1)+(z-y)(z+y+1)+(c-b)(z+y)+(z-y)(c+b),$$

i.e. if

$$by - ax = cz - by.$$

4. We next show that if Δ_a, Δ_b, Δ_c are in A.P., then numbers Δ_x, Δ_y, Δ_z (not necessarily rational) in A. P., can always be found, such that Δ_{a+x}, Δ_{b+y}, Δ_{c+z} are also in A. P.

Take $\mu = 1$, and let a, b, c correspond to the values α and β (not necessarily rational now) of λ_2 and λ_3 in §2.

Let x, y, z correspond to the values $\alpha + k$, $\beta + k$ of λ_2 and λ_3; then
$x = \{(\beta + k)^2 + 2\,(\alpha + k)\,(\beta + k) - (\alpha + k)^2 - 1\}/2 = a + 2\beta k + k^2$
$y = \{(\alpha + k)^2 + (\beta + k)^2 - 1\}/2 = b + (\alpha + \beta)\,k + k^2;$
$z = \{(\alpha + k)^2 + 2\,(\alpha + k)\,(\beta + k) - (\beta + k)^2 - 1\}/2 = c + 2\,\alpha k + k^2.$

After a little simplification, it can be shown that ax, bx, cz are in A. P. if

$$k^2 = 1.$$

Hence the required x, y, z are given by

$$a \pm 2\beta + 1, b \pm (\alpha + \beta) + 1, c \pm 2\alpha + 1.$$

EXAMPLE. Take $\alpha = 7, \beta = 4$;

then

$$a = 11, \quad b = 32, \quad c = 44,$$

and

$$x = 20, \quad y = 44, \quad z = 59,$$

or

$$x = 4, \quad y = 22, \quad z = 31.$$

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A note on the magic square of 9 cells

By M. N. KALE, *University College, London*

We may take after Chernick [J. CHERNICK : Solution of the general magic square, *American Math. Monthly*, 45, 1938, 172-175]

$k+a$	$k-a-b$	$k+b$
$k+b-a$	k	$k+a-b$
$k-b$	$k+a+b$	$k-a$

as the standard form of the magic square of 9 cells, with k, a and b positive. Each such form leads to four distinct squares obtained by reflection. The object of this note is to find the number of magic squares with distinct positive elements for a given k. For this we need find only the number of standard squares.

Since all the elements are positive and distinct, we must have

$$a+b<k, \tag{1}$$

$$\left.\begin{array}{l} a \neq b, \\ a \neq 2b, \\ 2a \neq b, \\ a>0,\ b>0 \end{array}\right\} \tag{2}$$

Now it can be easily shown that the number of solutions of

$$a+b = n \tag{3}$$

under the conditions (2) is

$$\left.\begin{array}{l} (n-1), \text{ if } (n,6)=1, \\ (n-2), \text{ if } (n,6)=2, \\ (n-3), \text{ if } (n,6)=3, \\ (n-4), \text{ if } (n,6)=6. \end{array}\right\} \tag{4}$$

Giving to n the values 1,2,3,..., $k-1$ in turn the total number of standard magic squares is readily seen to be

$$S = \frac{k(k-1)}{2} - \{1+2+3+2+1+4+1+2+3+2+1+4\ldots\text{to } (k-1) \text{ terms}\}$$

$$= \frac{k(k-1)}{2} - 13\left[\frac{k-1}{6}\right] - \lambda,$$

where $\lambda = 0, 1, 3, 6, 8, 9$ according as $k = 1, 2, 3, 4, 5, 0 \pmod 6$ and $[x]$ denotes the integral part of x as usual.

The total number of magic squares is evidently $4S$.

The author takes this opportunity to express his indebtedness to Dr. Hansraj Gupta who provided the solution to (3) under the conditions (2).

On the integration of the Darboux-Riccati equation for the general helix

By SULAIMAN KERAWALA, *Karachi University*

1. The integration of the Serret-Frenet equations can be reduced to the integration of a Riccati equation, known as the Darboux-Riccati equation. A suitable substitution reduces this equation to a second order linear equation. Recently, Pöschl [Theodor Pöschl, Sull' integrazione di Darboux-Riccati nella teoria delle curve a curvatura doppia, *Rend. Circ. Mat. Palermo* (2), 1, 88-91, 1952] has integrated this last equation in the case of the helix on the right circular cone and on the right circular cylinder. The purpose of the present note is to integrate the Darboux-Riccati equation directly in the case of the general helix.

2. If t, n, b refer to the direction-cosines of the tangent, normal and the binormal respectively of the space-curve, if ρ, τ denote its two radii of curvature, and if

$$\sigma = \frac{t + in}{1 - b}, \tag{1}$$

the Darboux-Riccati equation runs

$$\frac{d\sigma}{ds} = \frac{i}{2\tau}\left(\sigma^2 - \frac{2\tau}{\rho}\sigma - 1\right). \tag{2}$$

The substitution

$$\sigma = u + \tau/\rho \tag{3}$$

transforms (2) into

$$\frac{du}{ds} = \frac{i}{2\tau}u^2 - \left[\frac{d}{ds}\left(\frac{\tau}{\rho}\right) + \frac{i}{2\tau}\left(1 + \frac{\tau^2}{\rho^2}\right)\right] \tag{4}$$

3. In the case of the general helix, τ/ρ is a constant c. On writing $k^2 = 1 + c^2$ and $\psi = \int_0^s \frac{k\,ds}{\tau}$, (4) integrates to

$$\frac{\sigma - c}{k} = \frac{1 + Ae^{i\psi}}{1 - Ae^{i\psi}}, \tag{5}$$

where A is an arbitrary constant. If $\sigma = \sigma_0$ when $s = 0$, (5) assumes the form

$$\sigma = \frac{\{\sigma_0(k + c) + 1\} + e^{i\psi}\{\sigma_0(k - c) - 1\}}{(\sigma_0 + k - c) - e^{i\psi}(\sigma_0 - k - c)}. \tag{6}$$

Now let the moving trihedral at $s = 0$ coincide with the axes of reference, so that $t = (1, 0, 0)$, $n = (0, 1, 0)$, and $b = (0, 0, 1)$ at that point.

Case (i). σ is defined by the x-components of t, n, b, so that $\sigma_0 = 1$. (6) simplifies to

$$\sigma = \frac{(1 + c^2\cos\psi) - ick\sin\psi}{k^2 - c(1 - \cos\psi)}. \tag{7}$$

As $t_x^2 + n_x^2 + b_x^2 = 1$, it follows from (7) that

$$t_x = \frac{1 + c^2 \cos\psi}{k^2},\ n_x = -\frac{c \sin\psi}{k} \text{ and } b_x = \frac{c(1 - \cos\psi)}{k^2}. \quad (8)$$

CASE (ii). σ is defined by the y-components, so that $\sigma_0 = i$. (6) now simplifies to

$$\sigma = \frac{c \sin\psi + ik \cos\psi}{k + \sin\psi}. \quad (9)$$

As $t_y^2 + n_y^2 + b_y^2 = 1$, it follows from (9) that

$$t_y = \frac{c}{k} \sin\psi,\ n_y = \cos\psi, \text{ and } b_y = -\frac{\sin\psi}{k}. \quad (10)$$

CASE (iii). As the trihedral is right-handed, it follows from (8) and (10) that

$$t_z = \frac{c(1 - \cos\psi)}{k^2},\ n_z = \frac{\sin\psi}{k}, \text{ and } b_z = \frac{c^2 + \cos\psi}{k^2}. \quad (11)$$

(8), (10) and (11) complete the integration of the corresponding Serret-Frenet equations.

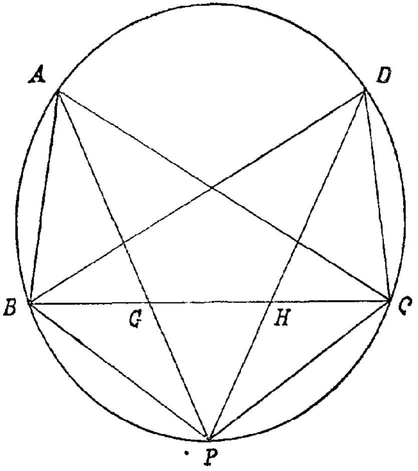
4. For curves which are not helices, the integration of (4) appears to present greater difficulties.

CLASSROOM NOTES

A case of congruence of triangles

By A. N. S. R. Aiyangar, *Thuckalay* (*Travancore-Cochin*)

The object of this note is to prove the theorem: *If the base, the vertical angle and the bisector of the vertical angle of one triangle are respectively equal to those of another, the two triangles are congruent.*



Let the two triangles be placed on the common base BC and let their vertices lie on the same side of it at A and D say.

Since $\angle A = \angle D$, the circum-circle of $\triangle ABC$ passes through D.

Let P be the mid-point of arc BC opposite to A. Join PB and PC, also join PA, and PD cutting BC at G and H respectively. AG and DH are now the bisectors of angles A and D and these are equal in length by hypothesis.

Since $\angle PBC = \angle PAC = \angle PAB$, i.e. $\angle BAG$,

therefore, PB is tangent to the circle through B, A, G.

Hence $PG.PA = PB^2$. Similarly $PH.PD = PC^2 = PB^2$.

Thus $PG.PA = PH.PD$, and $AGHD$ is cyclic.

But $AG = DH$, therefore $AD \parallel GH$ i.e. $AD \parallel BC$.

Hence arc $AB =$ arc DC, so that $AB = DC$, and arc $DAB =$ arc ADC, so that $BD = AC$.

It readily follows that the two triangles are congruent.

CORROLARY. *If the bisectors of two angles of a triangle are equal the sides opposite to them are also equal.*

Let BE and CF the bisectors of the base angles B and C of $\triangle ABC$ be equal and suppose BE and CF intersect at I. Then $\triangle$s ABE and ACF are congruent by the above theorem and $AB = AC$.

Bobbilier's theorem

By S. NARAYANAN, *Madura*

The pedal circle of a fixed point on a given fixed conic with respect to a variable triangle inscribed in the conic passes through a fixed point.

The general equation of a conic with the tangent and the normal to it at the given fixed point as the axes of coordinates is

$$ax^2 + xy + by^2 - gx = 0.$$

Any point on it may be denoted by

$$x = g/a + \lambda + b\lambda^2, \qquad y = \lambda g/a + \lambda + b\lambda^2,$$

λ being a variable parameter.

Let the parameters of any three variable points on the conic be λ_1, λ_2, λ_3.

The equation of the line joining λ_2 and λ_3 is

$$x(a - b\lambda_2\lambda_3) + y(1 + b\lambda_2 + b\lambda_3) - g = 0,$$

or briefly

$$xA_1 + yB_1 - g = 0.$$

The coordinates of the foot of the perpendicular from the origin to this line are given by

$$x = gA_1/(A_1^2 + B_1^2), \qquad y = gB_1/(A_1^2 + B_1^2).$$

The equation of the circle through the three feet of the perpendiculars is easily shown to be

$$\gamma(x^2 + y^2) = g(\alpha x + \beta y + g)$$

where

$$\alpha = b(1 + \Sigma\lambda_2\lambda_3) - 2a, \quad \beta = b(\lambda_1\lambda_2\lambda_3 - \Sigma\lambda_1) - 2,$$

and

$$\gamma = ab(1 + \Sigma\lambda_2\lambda_3) + b(\lambda_1\lambda_2\lambda_3 - \Sigma\lambda_1) - b^2\Sigma\lambda_2\lambda_3 - a^2 - 1.$$

Now to find the fixed point if any, that may lie on this circle, equate to zero the coefficients of $\Sigma\lambda_1$, $\Sigma\lambda_2\lambda_3$, $\lambda_1\lambda_2\lambda_3$ and the independent term, we have the three equations

$$x^2 + y^2 = gy, \qquad x^2 + y^2 = gx/(a - b)$$

and

$$x^2 + y^2 = \frac{gx(b - 2a) - 2gy + g^2}{ab - a^2 - 1}.$$

Solving we have

$$x/(a - b) = y = g/\{1 + (a - b)^2\}, \text{ which is the fixed point.}$$

In the case of a rectangular hyperbola, $a + b = 0$,

$$x = 2ag/(1 + 4a^2), \; y = g/(1 + 4a^2),$$

which is the centre of the hyperbola.

QUESTIONS AND SOLUTIONS

QN. 1835. (DEBA BRATA GHOSH). *If n is a positive integer prove that*

$$\int_0^{\pi/2} \theta \sin^{2n+1} \theta \, d\theta$$

$$= \frac{1}{2n+1} \left\{ 1 + \frac{2}{3} \cdot \frac{2n}{2n-1} + \frac{2.4}{3.5} \cdot \frac{(2n)(2n-2)}{(2n-1)(2n-3)} + \right.$$

$$\left. + \dots + \frac{2.4 \dots (2n)}{3.5 \dots (2n+1)} \cdot \frac{(2n)(2n-2) \dots 2}{(2n-1) \dots 1} \right\}.$$

SOLUTION BY B. S. GAI.

$$I_n = \int_0^{\pi/2} x \sin^{2n+1} x \, dx = \int_0^{\pi/2} \sin^{2n} x \, d(\sin x - x \cos x).$$

After a little reduction,

$$I_n = \frac{1}{(2n+1)^2} + \frac{2n}{2n+1} I_{n-1}.$$

Therefore

$$I_n = \frac{1}{(2n+1)^2} + \frac{2n}{2n+1} \cdot \frac{1}{(2n-1)^2} +$$

$$+ \frac{(2n)(2n-2)}{(2n+1)(2n-1)} \cdot \frac{1}{(2n-3)^2} + \dots +$$

$$+ \frac{(2n)(2n-2)(2n-4) \dots 2}{(2n+1)(2n-1)(2n-3) \dots 3} \cdot \frac{1}{1^2}$$

$$= \frac{2.4.6 \dots (2n)}{3.5.7 \dots (2n+1)} \left[a_n \cdot \frac{1}{2n+1} + a_{n-1} \cdot \frac{1}{2n-1} + \right.$$

$$\left. + a_{n-2} \cdot \frac{1}{2n-3} + \dots + a_1 \cdot \frac{1}{3} + 1 \right],$$

where $a_n = \dfrac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)}$

$$= \frac{2.4.6 \dots (2n)}{3.5.7 \dots (2n+1)} \left[\text{coefficient of } x^{2n+1} \text{ in the power series for } (\sin^{-1} x)(1-x^2)^{-1} \right],$$

since $\sin^{-1} x = \dfrac{x}{1} + a_1 \dfrac{x^3}{3} + a_2 \dfrac{x^5}{5} + \dots .$

Now

$$(\sin^{-1} x)(1-x^2)^{-1}$$

$$= \left[\frac{\sin^{-1} x}{\sqrt{(1-x^2)}}\right].(1-x^2)^{-\frac{1}{2}}$$

$$= \left[1.\frac{x}{1!} + 2^2.\frac{x^3}{3!} + 2^2.4^2.\frac{x^5}{5!} + \ldots + \right.$$

$$\left. + 2^2.4^2.6^2\ldots(2n)^2 \frac{x^{2n+1}}{(2n+1)!} + \ldots\right] \times$$

$$\times [1 + a_1 x^2 + a_2 x^4 + \ldots + a_n x^{2n} + \ldots]$$

$$= \left[x + \frac{2}{3}x^3 + \frac{2.4}{3.5}x^5 + \ldots + \frac{2.4.6\ldots(2n)}{3.5.7\ \ .(2n+1)}x^{2n+1} + \ldots\right] \times$$

$$\times [1 + a_1 x^2 + a_2 x^4 + \ldots].$$

Therefore

$$I_n = \frac{2.4.6\ldots(2n)}{3.5.7\ldots(2n+1)}\left[a_n + \frac{2}{3}a_{n-1} + \frac{2.4}{3.5}a_{n-2} + \right.$$

$$\left. + \ldots + \frac{2.4.6\ldots(2n)}{3.5.7\ldots(2n+1)}.1\right]$$

$$= \frac{1}{2n+1}\left[1 + \frac{2}{3}.\frac{2n}{2n-1} + \frac{2.4}{3.5}\frac{(2n)(2n-2)}{(2n-1)(2n-3)} + \ldots + \right.$$

$$\left. + \frac{2.4.6\ldots(2n)}{3.5.7\ldots(2n+1)}.\frac{(2n)(2n-2)(2n-4)\ldots 2}{(2n-1)(2n-3)(2n-5)\ldots 1}\right].$$

QN. 1837. (T. NARAYANA MOORTY). *If* $|\theta| \leqslant \pi/2$ *and* $x = 2\sin\theta$ *show that*

$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{(2!)^2 x^7}{7!} + \frac{(3!)^2 x^9}{9!} + \ldots = 8\theta\cos\theta + 2(2\theta^2 - 3)\sin\theta.$$

SOLUTION BY NARENDRA T. SHETH. Let the series on the L.H.S. be u. Then we have to show that

$$u = 8\theta\cos\theta + 2(2\theta^2 - 3)\sin\theta.$$

Differentiate both side w.r. to x then

$$\frac{du}{dx}=\frac{du/d\theta}{dx/d\theta}=\frac{8\cos\theta-8\theta\sin\theta+2\left[4\theta\sin\theta+(2\theta^2-3)\cos\theta\right]}{2\cos\theta}$$

$$=1+2\theta^2;$$

$$=1+2\left[\sin^{-1}\frac{x}{2}\right]^2. \tag{1}$$

Now, by Taylor's theorem on expansion of functions, it can be easily shown that for $-1\leqslant x\leqslant 1$,

$$\left(\sin^{-1}\frac{x}{2}\right)^2=2\left[\frac{1}{2}\left(\frac{x}{2}\right)^2+\frac{2}{3.4}\left(\frac{x}{2}\right)^4+\frac{2.4}{3.5.6}\left(\frac{x}{2}\right)^6+\right.$$
$$\left.+\frac{2.4.6}{3.5.7.8}\left(\frac{x}{2}\right)^8+\dots\right]$$

Therefore substituting in (1) we get

$$\frac{du}{dx}=1+4\left[\frac{x^2}{8}+\frac{x^4}{96}+\frac{x^6}{720}+\dots\right]$$
$$=1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{180}+\frac{x^8}{35.128}+\dots.$$

Integrating both sides w.r. to x, we get

$$u=x+\frac{x^3}{3!}+\frac{x^5}{5!}+(2!)^2\frac{x^7}{7!}+(3!)^2\frac{x^9}{9!}+\dots.$$

The term by term integration is valid as the series is uniformly convergent. (The constant $c=0$).

Also solved by C. B. Rathie.

QN. 1839. (V. THÉBAULT). *In an orthocentric tetrahedron inscribed to a sphere (O, R), if the angle between the inscribed sphere (I, r) and the second twelve-points-sphere is θ, then*

$$\overline{OI}^2=R^2-3r^2-2Rr\cos\theta,$$

a relation that may be compared to that of Euler in a triangle.

SOLUTION BY PROPOSER. Consider the following generalization. If (H) is the power of the orthocenter H of an orthocentric tetrahedron $T\equiv ABCD$, with respect to the circumsphere, if Ω is the

center of the second twelve-points sphere $(\Omega, R/3)$ and if (ω, σ) is the principal sphere of the revolution quadric (Q) inscribed in T and having any point P as focus, then, according to Faure's theorem, the sum of the squares of the half-axes of (Q) equals the power of ω with respect to the sphere conjugated to T, the square of the radius of that sphere being (H). Hence, as

$$\overline{OH}^2 = R^2 + 3(H), \tag{1}$$

we have

$$\overline{\omega H}^2 = 3\sigma^2 + (H) - \overline{2\omega P}^2. \tag{2}$$

From (1) and (2), we derive

$$\overline{OH}^2 - 3\,\overline{\omega H}^2 = R^2 - 9\,\sigma^2 + \overline{6\omega P}^2.$$

But $O\Omega = 2\Omega H$; hence

$$\overline{\omega O}^2 = 3\,\overline{\omega\Omega}^2 + (2/3)\ R^2 - 6\,\sigma^2 + 4\,\overline{\omega P}^2. \tag{3}$$

If the sphere (ω, σ) cuts the sphere $(\Omega, R/3)$, i.e. if

$$(\sigma - R/3) < \omega\Omega < (\sigma + R/3),$$

and if θ is the angle formed by both spheres, then (3) leads to

$$\overline{O\omega}^2 = R^2 - 3\sigma^2 - 2R\sigma\cos\theta + 4\,\overline{\omega\rho}^2, \tag{4}$$

as

$$\overline{\omega\Omega}^2 = R^2/9 + \sigma - 2(R/3)\cos\theta.$$

SPECIAL CASE. *When $P \equiv I$ and when the sphere (I, r) cuts the sphere $(\Omega, R/3)$, i.e. when*

$$(R - 3r)\,(R + r) < \overline{OI}^2 < (R + 3r)\,(R - r),$$

then

$$\overline{OI}^2 = R^2 - 3r^2 - 2R^2\cos\theta.$$

QN. 1840. (V. THÉBAULT). *Find numeration-systems of basis B, in which there are four-figure numbers $abcd$, such that the sum $a + d + ba$ equals the square of B and that the number cd equals the square of the difference $|d - a|$ of the end figures. Apply to $B = 10$, $B = 65$.*

HINTS FOR SOLUTION BY PROPOSER. As

$$B\,b + a + a + d = B^2, \tag{1}$$

$$B\,c + b = |d - a|^2, \tag{2}$$

we have

$$2a + d = M.B,$$

or

$$2a + d = B, \qquad 2a + d = 2B.$$

(1). $2a + d = B$. Then $B^2 = B\,b + B$, or $b = B - 1$,

$$B(c + 1) = (d - a)^2 + 1\,; \tag{3}$$

it follows that B and $c + 1$ are both the sum of two squares.

(2). $2a + d = 2B$. When d is even and $B^2 = B\,b + 2B$, or $b = B - 2$; further

$$B(c + 1) = (d - a)^2 + 2,$$

and B and $c + 1$ are both the sum of three squares.

For instance, in case (1), the smallest value of c is $c = 0$ and

$$N = \overline{abcd} = a\ \overline{B - 1}\ 0\ d.$$

From (3), we derive $B - 1 = (d - a)^2$; $d - a = \pm\sqrt{(B - 1)}$, and if $d > a$, $d - a = \sqrt{(B - 1)}$, or in integers,

$$d = \tfrac{1}{3}\,(B + 2\sqrt{(B - 1)}),\ a = \tfrac{1}{3}\,(B - \sqrt{(B - 1)}),$$

$$N = \tfrac{1}{3}\,\overline{(B - \sqrt{(B - 1)})}\ \overline{B - 1}\ \ 0\ \ \overline{\tfrac{1}{3}\,(B + 2\sqrt{(B - 1)})}.$$

The values of B must correspond to integers : for instance, if $B = 65$,

$$N = \overline{19}\ \overline{64}\ \ 0\ \ \overline{27}.$$

QN. 1841. (T. NARAYANA MOORTY). *If* a_t $(1 \leqslant t \leqslant r)$ *and* b_t $(1 \leqslant t \leqslant s)$ *are any* $r + s$ *real quantities* $(r \geqslant s)$, *prove that*

$$r\left[\sum_1^r a_t^2 . \sum_1^s b_t^2 - \left(\sum_1^s a_t b_t\right)^2\right] \geqslant \left(\sum_1^r a_t^2\right)\left(\sum_1^s b_t\right)^2 +$$

$$+ \left(\sum_1^r a_t\right)^2\left(\sum_1^s b_t^2\right) - 2\left(\sum_1^r a_t\right)\left(\sum_1^s b_t\right)\left(\sum_1^s a_t b_t\right).$$

SOLUTION BY PROPOSER. Set $b_t = 0$ $(S < t \leqslant r)$, so that a_t, b_t are both defined for $1 \leqslant t \leqslant r$. Now a_t can be considered as the r values of one variable x and b_t as the corresponding r values of another variable y. Then

$$\bar{x} = \sum_{1}^{r} a_t/r,\ \bar{y} = \sum^{s} b_t/r,$$

since the other b's are zeros.

With the convention that Σ refers to $\sum_{1}^{r}$ when referring to a's and to $\sum_{1}^{s}$ when referring to a's and b's or only b's we have

$$\sigma_x^2 = \frac{1}{r^2}\left[r\ \Sigma a^2 - (\Sigma a)^2\right];\ \sigma_y^2 = \frac{1}{-r^2}\left[r\ \Sigma b^2 - (\Sigma b)^2\right]$$

and

$$r^2.\ r_{xy}\ \sigma_x\ \sigma_y = r\ \Sigma ab - \Sigma a.\ \Sigma b.$$

Since $|r_{xy}| \leqslant 1$, we have

$$(r\ \Sigma ab - \Sigma a\ \Sigma b)^2 = r^4.\ r_{xy}^2\ \sigma_x^2\ \sigma_y^2$$

$$\leqslant r^4.\ \frac{1}{r^2}\left[r\ \Sigma a^2 - (\Sigma a)^2\right].\ \frac{1}{r^2}\left[r.\ \Sigma b^2 - (\Sigma b)^2\right],$$

$$r^2(\Sigma ab)^2 + (\Sigma a)^2\ (\Sigma b)^2 - 2r\ \Sigma ab\ \Sigma a\ \Sigma b \leqslant r^2\ \Sigma a^2\ \Sigma b^2 + (\Sigma a)^2\ (\Sigma b)^2 -$$
$$- r\ [\Sigma a^2\ (\Sigma b)^2 + \Sigma b^2\ (\Sigma a)^2].$$

On transposition we get the required inequality.

Question for solution

QN. 1842. (ALFRED MOESSNER). *Give a solution of the equation $A^5 + B^5 = C^5 + D^5$ in rational numbers or prove that the equation has no solution in rational numbers.*

[A prize of 50 German marks will be awarded to the first correct solution received. The solution should be sent by registered air mail to Dr. Alfred Moessner, 13a, Gunzenhausen, (Germany-Bayern), Wiesenstrasse, 15.]

NEWS AND NOTICES

The following members have been admitted as life members of the Society : N. S. Nagendranath and T. R. Raghava Sastry.

The following persons have been elected ordinary members of the Society : E. C. George, M. N. Korgaokar, Marakatha Krishnan, M. S. Narasimhan, Naresh Kumar, R. Ramachandran, S. Ramakrishnan, P. K. Raman and C. S. Seshadri.

The Government of India has deputed Professor K. Chandrasekharan as the Indian delegate to the International Mathematical Union and Professor K. Chandrasekharan and Professor B. R. Seth as delegates to the International Congress of Mathematicians (September 2-9, 1954), Amsterdam.

M. S. Ramanujan has been awarded the M.Sc. degree by the Annamalai University for his thesis on ' General summation processes with reference to Hausdorff methods '.

Professor G. L. Chandratreya has been appointed Principal, Willingdon College, Sangli.

Dr. U. N. Singh has been appointed Reader in Mathematics, Muslim University, Aligarh.

The following members of the Society received honours on Independence Day, 1954 : Professor C. V. Raman (Bharat Ratna), Professor H. J. Bhabha and Professor K. S. Krishnan (Padma Vibhushan). We offer them our congratulations.

We regret to record the death of Professor A. N. Singh, formerly Head of the Department of Mathematics and Statistics, Lucknow University. He was a life member of the Society and served on the Council of the Society for a number of years and was Local Secretary of the Lucknow Conference of the Society in 1937. His books, ' The theory and construction of non-differentiable functions ' and ' A history of Hindu mathematics ' (in collaboration with B. B. Datta) are well known. We offer our condolences to the bereaved family.

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GEGENBAUER TRANSFORMS

By S. K. LAKSHMANARAO

1. Introduction. After Doetsch [2] extended the theory of Fourier transforms to finite ranges of the variable, several others have made studies with finite integral transforms. Sneddon [5] employed the finite Hankel transform for the solution of some initial and boundary value problems in the theory of elastic vibrations. Tranter [6] defined the finite integral transform with the kernel $P_n(x)$, called it a Legendre transform and used it in some applications. Recently Scott [4] introduced a finite integral transform of considerable generality by choosing the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ for the kernel and with the choice of $\alpha = \beta = 0$ this reduces to the Legendre transform of [6].

The derivation of a convolution property is of much importance for the full development of the operational calculus based on any integral transform as it enables us to expand substantially the tables of transforms. The convolution property for finite Fourier transforms is derived by Doetsch [2] and for Legendre transforms by Churchill and Dolph in a recent paper [1]. A similar derivation for the Jacobi transform is desirable, but is not possible as there is no simple addition formula for $P_n^{(\alpha,\beta)}(x)$.

In the present paper we discuss an integral transform over the range $(-1, 1)$ with $(1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x)$ for the kernel, where $C_n^\nu(x)$ is the Gegenbauer polynomial, and call it a Gegenbauer transform. This is certainly less general than the Jacobi transform of [4] but the existence of an addition formula for $C_n^\nu(x)$ facilitates the derivation of the convolution property. The actual derivation of the convolution property is given below in a manner completely similar to that in [1].

2. The Gegenbauer transform. If $F(x)$ denotes a continuous function on the interval $-1 \leqslant x \leqslant 1$. its Gegenbauer transform is defined by the relation

$$f^{\nu}(n) = TF(x)$$
$$= \int_{-1}^{+1} F(x)\,(1-x^2)^{\nu-\frac{1}{2}}\, C_n^{\nu}(x)dx,\ \nu > -\tfrac{1}{2},\ n = 0, 1, 2, \ldots, \quad (1)$$

where the Gegenbauer polynomial $C_n^{\nu}(x)$ is a solution of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - (2\nu+1)\,x\frac{dy}{dx} + n\,(n+2\nu)\,y = 0. \quad (2)$$

We obtain after some calculation the following fundamental operational property of this transform :

$$T\left\{(1-x^2)^{-(\nu-\frac{1}{2})}\frac{d}{dx}\left[(1-x^2)^{\nu+\frac{1}{2}}\frac{dF(x)}{dx}\right]\right\} = -\,n(n+2\nu)\,f^{\nu}(n). \quad (3)$$

This relation suggests that the Gegenbauer transforms facilitate the solution of boundary value problems involving the differential form

$$(1-x^2)^{-(\nu-\frac{1}{2})}\frac{d}{dx}\left[(1-x^2)^{\nu+\frac{1}{2}}\frac{dF(x)}{dx}\right].$$

The inverse transform $T^{-1}\{f^{\nu}(n)\}$ denotes the function defined over $(-1, 1)$ whose Gegenbauer transform is $f^{\nu}(n)$. To invert the equation (1) we may put $F(x)$ in the form

$$F(x) = \sum_{n=0}^{\infty} c_n C_n^{\nu}(x). \quad (4)$$

Invoking the orthogonality relations for $C_n^{\nu}(x)$ [3], viz.

$$\int_{-1}^{+1} C_n^{\nu}(x)\, C_m^{\nu}(x)\,(1-x^2)^{\nu-\frac{1}{2}}\,dx = 0, \qquad (m \neq n),$$
$$\int_{-1}^{+1} \{C_n^{\nu}(x)\}^2\,(1-x^2)^{\nu-\frac{1}{2}}\,dx = \frac{2^{1-2\nu}\pi\,\Gamma(n+2\nu)}{n!\,(\nu+n)\,[\Gamma(\nu)]^2},$$

and after a formal calculation, we obtain

$$F(x) = \sum_{n=0}^{\infty} \frac{n!\,(n+\nu)\,[\Gamma(\nu)]^2\,2^{2\nu-1}}{\pi\,\Gamma(n+2\nu)} f^{\nu}(n)\,C_n^{\nu}(x),\ -1 < x < 1. \quad (5)$$

The relations (1) and (3) enable us to replace the known and unknown functions in a problem by their Gegenbauer transforms and then

from (5) we can have an expansion for the unknown function. To extend this operational calculus somewhat, we now derive the convolution property.

3. Derivation of the convolution property. $F(x)$ and $G(x)$ are continuous functions on the interval $-1 \leqslant x \leqslant 1$, and $f^\nu(n)$, $g^\nu(n)$ are their Gegenbauer transforms. Then

$$f^\nu(n)\, g^\nu(n) = \int_{-1}^{+1} F(x)\,(1-x^2)^{\nu-\frac{1}{2}}\, C_n^\nu(x)dx \times \\ \times \int_{-1}^{+1} G(x)\,(1-x^2)^{\nu-\frac{1}{2}}\, C_n^\nu(x)dx$$

$$= \int_0^\pi F(\cos\eta)\,(\sin\eta)^{2\nu}\, C_n^\nu(\cos\eta)\,d\eta \times \\ \times \int_0^\pi G(\cos\xi)\,(\sin\xi)^{2\nu}\, C_n^\nu(\cos\xi)d\xi$$

$$= \int_0^\pi F(\cos\eta)\,(\sin\eta)^{2\nu}\,d\eta \times \\ \times \int_0^\pi G(\cos\xi)\,(\sin\xi)^{2\nu}\, C_n^\nu(\cos\xi)\, C_n^\nu(\cos\eta)d\xi.$$

Using the addition formula [3]

$$C_n^\nu(\cos\xi)\, C_n^\nu(\cos\eta) = A\int_0^\pi C_n^\nu(\cos\xi\cos\eta + \sin\xi\sin\eta\cos\alpha) \times \\ \times (\sin\alpha)^{2\nu-1}\,d\alpha, \tag{6}$$

where

$$A = \frac{\Gamma(n+2\nu)\; 2^{1-2\nu}}{n!\;[(\Gamma(\nu)]^2}, \tag{7}$$

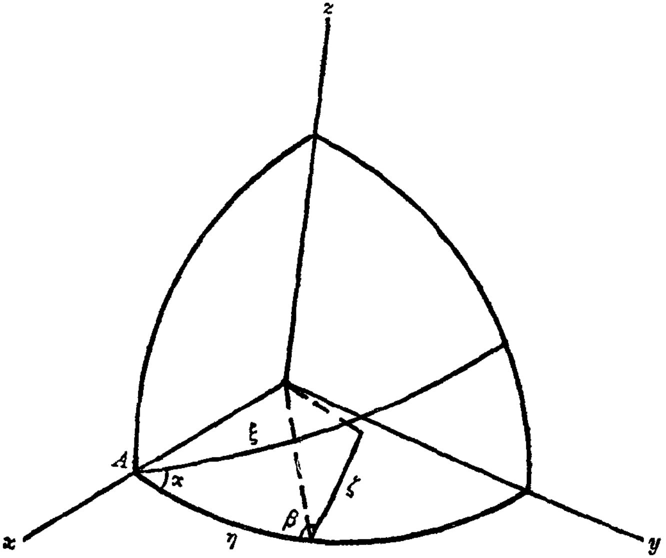
we have

$$f^\nu(n)\, g^\nu(n)$$

$$= A\int_0^\pi F(\cos\eta)\,(\sin\eta)^{2\nu}\,d\eta \int_0^\pi G(\cos\xi)\,(\sin\xi)^{2\nu}\,d\xi \times \\ \times \int_0^\pi C_n^\nu(\cos\xi\cos\eta + \sin\xi\sin\eta\cos\alpha)\,(\sin\alpha)^{2\nu-1}\,d\alpha$$

$$= A\int_0^\pi F(\cos\eta)\,(\sin\eta)^{2\nu}\,d\eta \times \\ \times \Big[\int_0^\pi\int_0^\pi G(\cos\xi)\, C_n^\nu(\cos\xi\cos\eta + \sin\xi\sin\eta\cos\alpha)\,(\sin\xi)^{2\nu} \times \\ \times (\sin\alpha)^{2\nu-1}\,d\xi\,d\alpha\Big].$$

As in [1] we can interpret the iterated integral in the above as a surface integral over the unit hemisphere $X^2 + Y^2 + Z^2 = 1, Z > 0$, with ξ, η, ζ and α as explained in the figure.



Now changing the variables from (α, ξ) to (β, ζ) in the integral, through the relations

$$\cos \xi = \cos \eta \cos \zeta + \sin \eta \sin \zeta \cos \beta,$$

$$\sin \alpha = \sin \zeta \frac{\sin \beta}{\sin \xi},$$

we have after the substitution

$$\int_0^\pi \int_0^\pi G(\cos \xi)\, C_n^\nu(\cos \xi \cos \eta + \sin \xi \sin \eta \cos \alpha) \times$$

$$\times (\sin \xi)^{2\nu} (\sin \alpha)^{2\nu - 1}\, d\xi\, d\alpha$$

$$= \int_0^\pi \int_0^\pi G(\cos \eta \cos \zeta + \sin \eta \sin \zeta \cos \beta) \times$$

$$\times C_n^\nu(\cos \zeta)\, (\sin \zeta)^{2\nu} (\sin \beta)^{2\nu - 1}\, d\beta\, d\zeta,$$

$$f^{\nu}(n)\, g^{\nu}(n) = A \int_0^{\pi} F(\cos\eta)\, (\sin\eta)^{2\nu}\, d\eta \times$$
$$\times \int_0^{\pi}\int_0^{\pi} G(\cos\eta\cos\zeta + \sin\eta\sin\zeta\cos\beta) \times$$
$$\times\, C_n^{\nu}(\cos\zeta)\, (\sin\zeta)^{2\nu}\, (\sin\beta)^{2\nu-1}\, d\beta\, d\zeta$$
$$= A \int_0^{\pi} C_n^{\nu}(\cos\zeta)\, (\sin\zeta)^{2\nu}\, d\zeta \times$$
$$\times \int_0^{\pi}\int_0^{\pi} F(\cos\eta)\, (\sin\eta)^{2\nu}\, G(\cos\eta\cos\zeta + \sin\eta\sin\zeta\cos\beta) \times$$
$$\times\, (\sin\beta)^{2\nu-1}\, d\eta\, d\beta.$$

Let

$$H(\cos\zeta) = A \int_0^{\pi}\int_0^{\pi} F(\cos\eta)\, (\sin\eta)^{2\nu}\, G(\cos\eta\cos\zeta + \sin\eta\sin\zeta\cos\beta) \times$$
$$\times\, (\sin\beta)^{2\nu-1}\, d\eta\, d\beta. \quad (8)$$

Then

$$h^{\nu}(n) = T\, H(x) = f^{\nu}(n)\, g^{\nu}(n), \quad (9)$$

so that

$$H(x) = T^{-1}\, h^{\nu}(n) = F(x) * G(x), \quad (10)$$

is the convolution of $F(x)$ and $G(x)$.

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MATHEMATICAL NOTES

A problem on moments

By M. K. SINGAL and P. L. BHATNAGAR

1. Introduction. It is well known that the distance of the centre of gravity of a uniform plane triangular lamina from a fixed line lying in the plane of the lamina is given by

$$\bar{z} = \frac{z_1 + z_2 + z_3}{3}, \tag{1.1}$$

z_1, z_2, z_3 being the distances of the corner of the lamina from the fixed line.

(1.1) can also be written as

$$\bar{z} = \frac{A_1}{A_0}, \tag{1.2}$$

where A_n denotes the arithmetic mean of the different homogeneous products of the nth degree, formed of z_1, z_2, z_3.

The object of the present note is to generalize the formula (1.2) to n dimensions, the density at any point varying as the nth power of the distance from a hyper-plane, and to study some particular cases.

2. Let ξ_{r+1}, $r = 0, 1, 2, \ldots n$ be the homogeneous co-ordinates of a point in an Euclidean space E_n of n-dimensions, such that

$$\sum_{r=0}^{r=n} \xi_{r+1} = 1. \tag{2.1}$$

Let the interior of a simplex S_{n+1} imbedded in E_n be defined by the inequalities

$$\xi_{r+1} > 0, \quad r = 0, 1, 2, \ldots, n, \tag{2.2}$$

and let $u_1, u_2, \ldots u_{n+1}$ be the distances of the corners of the simplex from a hyper-plane E_{n-1}.

The distance of any point of the simplex from the space E_{n-1} is then given by

$$\sum_{r=0}^{r=n} u_{r+1}\, \xi_{r+1}. \tag{2.3}$$

If the density at any point varies as the pth power of the distance of the point from the hyper-plane E_{n-1}. the distance $\bar{z}$ of the C.G. of the simplex from E_{n-1} is given by

$$\bar{z} = \frac{\int (\sum_0^n u_{r+1}\, \xi_{r+1})^{p+1}\, dv}{\int (\sum_0^n u_{r+1}\, \xi_{r+1})^{p}\, dv}, \tag{2.4}$$

where the integrals are to be evaluated for the region defined by $\xi_{r+1} > 0$, $\Sigma\, \xi_{r+1} = 1$, $r = 0, 1, 2, \ldots, n$, dv being an element of volume.

Using rectangular co-ordinates x_s, $s = 0, 1, 2, \ldots, n$,

$$dv = dx_1\, dx_2 \ldots dx_n. \tag{2.5}$$

Since x's must be linear functions of n of the ξ's, say ξ_r, $r = 1, 2, \ldots, n$, we have

$$dv = C\, d\xi_1\, d\xi_2 \ldots d\xi_n, \tag{2.6}$$

where C depends on the relation between the two systems of co-ordinates the x's and the ξ's.

(2.4) then gives

$$\left.\begin{array}{c} \dfrac{\iint \cdots \int (\sum_0^n u_{r+1}\, \xi_{r+1})^{p+1}\, d\xi_1\, d\xi_2 \ldots d\xi_n}{\iint \cdots \int (\sum_0^n u_{r+1}\, \xi_{r+1})^{v}\, d\xi_1\, d\xi_2 \ldots d\xi_n}, \\ \xi_{r+1} > 0, \Sigma\, \xi_{r+1} = 1, r = 0, 1, \ldots, n \end{array}\right\}. \tag{2.7}$$

Now

$$\iint \cdots \int (\sum_0^n u_{r+1}\, \xi_{r+1})^{p+1}\, d\xi_1 \ldots d\xi_n$$

$$= \iint \cdots \int \sum \left\{ \frac{(p+1)!}{(a_1)! \ldots (a_{n+1})!} (u_1^{a_1}\, u_2^{a_2} \ldots u_{n+1}^{a_{n+1}}) \right\} d\xi_1 \ldots d\xi_n, \tag{2.8}$$

where $a_1, \ldots, a_{n+1} \geqslant 0$, $a_1 + a_2 + \ldots + a_{n+1} = p + 1$.

By Dirichlet's formula, if

$$\phi(a_1, a_2, \dots, a_{n+1}) = \iint \cdots \int \xi_1^{a_1} \xi_2^{a_2} \dots \xi_{n+1}^{a_{n+1}} \; d\xi_1 \, d\xi_2 \dots d\xi_n,$$

$$\xi_{r+1} > 0, \Sigma \, \xi_{r+1} = 1, r = 0, 1, \dots, n,$$

$$\phi(a_1, a_2, \dots, a_{n+1}) = \frac{\Gamma(a_1 + 1)\,\Gamma(a_2 + 1) \dots \Gamma(a_{n+1} + 1)}{\Gamma(a_1 + a_2 + a_3 + \dots + a_{n+1} + n + 1)}. \qquad (2.9)$$

By virtue of (2.9), (2.8) yields

$$\iint \cdots \int \Big(\sum u_{r+1} \, \xi_{r+1} \Big)^{p+1} d\xi_1 \, d\xi_2 \dots d\xi_n$$

$$= \sum_{a_1, \dots, a_{n+1}} \frac{(p+1)!}{(a_1)!\,(a_2)! \dots (a_{n+1})!} \cdot \frac{\Gamma(a_1 + 1)\,\Gamma(a_2 + 1) \dots \Gamma(a_{n+1} + 1)}{\Gamma(a_1 + a_2 + \dots + a_{n+1} + n + 1)} \times$$

$$\times \, u_1^{a_1} u_2^{a_2} \dots u_{n+1}^{a_{n+1}}$$

$$= \sum \frac{(p+1)!}{(p+1+n)!} u_1^{a_1} u_2^{a_2} \dots u_{n+1}^{a_{n+1}},$$

$$= \frac{(p+1)!}{(p+1+n)!} S_{n+1,p+1}, \qquad (2.10)$$

$a_1, a_2, \dots, a_{n+1}$ being positive integers (or zero) satisfying the relation $\sum_0^n a_{r+1} = p + 1$, where $S_{n+1,p+1}$ is the sum of all the different homogeneous products of $u_1, u_2, \dots, u_{n+1}$ of the $(p + 1)$th degree We shall denote it by

$$A_{n+1,p+1}, \qquad (2.11)$$

which, in our notation, stands for the arithmetic mean of all the different homogeneous products of $u_1, u_2, \dots, u_{n+1}$ of the $(p + 1)$th degree.

Using (2.10) and (2.11), (2.7) yields

$$\bar{z} = \frac{\dfrac{(p+1)!}{(p+1+n)!} S_{n+1,\,p+1}}{\dfrac{p!}{(p+n)!} S_{n+1,\,p}}$$

$$= (p + 1) \, S_{n+1,\,p+1}/(p + 1 + n) \, S_{n+1,\,p}, \qquad (2.12)$$

$$= A_{n+1,\,p+1}/A_{n+1,\,p}, \qquad (2.13)$$

which is a generalization of (1.2) to n-dimensions.

From the above it can be readily seen that

$$k^2 = \frac{(p+2)(p+1)}{(p+2+n)(p+1+n)} \frac{S_{n+1,\,p+2}}{S_{n+1,\,p}}, \tag{2.14}$$

$$= \frac{A_{n+1,\,p+2}}{A_{n+1,\,p}}, \tag{2.15}$$

are the expressions for the square of the radius of gyration of the simplex with respect to E_{n-1} when the density at any point varies as the pth power of the distance of the point from E_{n-1}.

3. Since $S_{n+1,\,p+1}$ is a homogeneous function of the $(p+1)$th degree in the $(n+1)$ variables $u_1, u_2, \ldots, u_{n+1}$, using Euler's theorem, we have

$$u_1 \frac{\partial S_{n+1,\,p+1}}{\partial u_1} + u_2 \frac{\partial S_{n+1,\,p+1}}{\partial u_2} + \ldots + u_{n+1} \frac{\partial S_{n+1,\,p+1}}{\partial u_{n+1}} = (p+1)\, S_{n+1,\,p+1}. \tag{3.1}$$

Also

$$\begin{aligned} \frac{\partial S_{n+1,\,p+1}}{\partial u_1} &= \sum_{a_1,\ldots,a_{n+1}} a_1\, u_1^{a_1-1}\, u_2^{a_2} \ldots u_{n+1}^{a_{n+1}}, \\ \frac{\partial S_{n+1,\,p+1}}{\partial u_2} &= \sum_{a_1,\ldots,a_{n+1}} a_2\, u_1^{a_1}\, u_2^{a_2-1} \ldots u_{n+1}^{a_{n+1}}, \ldots \\ \frac{\partial S_{n+1,\,p+1}}{\partial u_{n+1}} &= \sum_{a_1,\ldots,a_{n+1}} a_{n+1}\, u_1^{a_1}\, u_2^{a_2} \ldots u_{p+1}^{a_{n+1}-1}, \end{aligned} \tag{3.2}$$

where $a_1, a_2, \ldots, a_{n+1} \geqslant 0$, $a_1 + a_2 + \ldots + a_{n+1} = p+1$.

Adding up the equations in (3.2), we have

$$\sum_{r=0}^{r=n} \frac{\partial S_{n+1,\,p+1}}{\partial u_{r+1}} = \sum_{a_1,\ldots,a_{n+1}} (a_1\, u_1^{a_1-1}\, u_2^{a_2} \ldots u_{n+1}^{a_{n+1}} + a_2\, u_1^{a_1} u_2^{a_2-1} \ldots u_{n+1}^{a_{n+1}} + \\ + \ldots + a_{n+1}\, u_1^{a_1}\, u_2^{a_2} \ldots u_{n+1}^{a_{n+1}-1}),$$

$$a_{r+1} \geqslant 0,\ \Sigma\, a_{r+1} = p+1,\ r = 0, 1, \ldots n,$$

$$= \sum_{\beta_1,\ldots\beta_{n+1}} (a_1 + a_2 + \ldots + a_{n+1} + n)\, u_1^{\beta_1}\, u_2^{\beta_2} \ldots u_{n+1}^{\beta_{n+1}}$$

$$= (p+1+n)\, S_{n+1,p}\,; \tag{3.3}$$

where $$\beta_{r+1} \geqslant 0, \sum_{0}^{n} \beta_{r+1} = p.$$

Using (3.1) and (3.3), (2.12) yields

$$\bar{z} = \frac{\sum_{r=0}^{r=n} u_{r+1} \frac{\partial S_{n+1, p+1}}{\partial u_{r+1}}}{\sum_{r=0}^{r=n} \frac{\partial S_{n+1, p+1}}{\partial u_{r+1}}}. \tag{3.4}$$

From (3.4) it follows that *the distance from a hyper-plane E_{n-1}, of the C. G. of a simplex, the density at any point of which varies as the pth power of the distance of the point from E_{n-1}, is the same as that of the system of particles of masses proportional to*

$$\frac{\partial S_{n+1, p+1}}{\partial u_{r+1}}, \quad r = 0, 1, 2, \ldots, n,$$

placed at the corners of the simplex, u_{r+1}, $r = 0, 1, 2, \ldots, n$ being the distances of the corners of the simplex from E_{n-1}.

4. Particular Cases

(a) Putting $n = 3$ in the result arrived at in §3 we find that (i) the distance of the centre of gravity of a tetrahedron from a plane, the density at any point of the tetrahedron varying as the pth power of the distance of the point from the plane, is the same as that of particles of masses proportional to $\left(\frac{\partial S_{4,p+1}}{\partial u_1}, \frac{\partial S_{4,p+1}}{\partial u_2}, \frac{\partial S_{4,p+1}}{\partial u_3}, \frac{\partial S_{4,p+1}}{\partial u_4} \right)$ placed at the vertices of the tetrahedron; u_1, u_2, u_3, u_4, being the distance of the vertices of the tetrahedron from the plane.

(ii) Putting $p = 0$ in the above we find that the distance of the centre of gravity of a tetrahedron from a plane is the same as that of particles of masses proportional to $\left(\frac{\partial S_4}{\partial u_1}, \frac{\partial S_4}{\partial u_2}, \frac{\partial S_4}{\partial u_3}, \frac{\partial S_4}{\partial u_4} \right)$ placed at the vertices of the tetrahedron, u_1, u_2, u_3, u_4 being the distances of the vertices of the tetrahedron from the plane, where

$S_4 = u_1 + u_2 + u_3 + u_4$, showing that the distance of the C. G. from the plane is $\dfrac{u_1 + u_2 + u_3 + u_4}{4}$.

(iii) Putting $n = 3$, $p = 0$ in (2.14) and (2.15) we find that for a tetrahedron,

$$k^2 = \frac{1}{10}\frac{S_{4,2}}{S_{4,0}} = \frac{1}{10}\frac{u_1{}^2+u_2{}^2+u_3{}^2+u_4{}^2+(u_1+u_2)(u_3+u_4)+u_1u_2+u_3u_4}{u_1+u_2+u_3+u_4}.$$

(b) Putting $n=2$ in the results obtained in §2 above we find that (i) the distance of the centre of gravity of a plane triangular lamina from a fixed line the density at any point of which varies as the pth power of the distance from the fixed line, is given by

$$\bar{z} = \frac{p+1}{p+3}\frac{S_{3,p+1}}{S_{3,p}};$$

(ii) the square of radius of gyration of a plane triangular lamina with respect to any straight line lying in the plane of the lamina the density at any point of which varies as the pth power of the distance from the straight line, is given by

$$k^2 = \frac{(p+1)(p+2)}{(p+3)(p+4)}\frac{S_{3,p+2}}{S_{3,p}}.$$

(iii) It can be readily seen that the distance of the centre of pressure of a triangular lamina immersed in a fluid, from a line lying in the plane of the lamina, the density of the fluid, at any point varying as the pth power of the distance of the point from the line, is given by

$$\bar{z} = \frac{p+2}{p+4}\frac{S_{3\ p+2}}{S_{3,p+1}}$$
$$= A_{3,\ p+2}/A_{3,\ p+1}.$$

(iv) Putting $p = 0$ in the results (i), (ii), (iii) above we find that the distance of the C. G. of a triangular lamina from a straight line lying in the plane of the lamina is given by

$$\bar{z} = \frac{1}{3}\frac{S_{3,1}}{S_{3,0}} = \frac{u_1+u_2+u_3}{3};$$

(v) the square of the radius of gyration of a plane triangular lamina with respect to any straight line lying in the plane of the lamina is given by

$$k^2 = \frac{1}{6}\frac{S_{3,2}}{S_{3,0}} = \frac{1}{6}\left(u_1^2 + u_2^2 + u_3^2 + u_2 u_3 + u_3 u_1 + u_1 u_2\right);$$

(vi) the distance of the centre of pressure of a triangular lamina immersed in a homogeneous fluid, from the line of intersection of the effective surface and the plane of the lamina is given by

$$\bar{z} = \frac{1}{2}\frac{S_{3,2}}{S_{3,1}} = \frac{1}{2}\,\frac{u_1^2 + u_2^2 + u_3^2 + u_2 u_3 + u_3 u_1 + u_1 u_2}{u_1 + u_2 + u_3},$$

u_1, u_2, u_3 being the distances of the vertices from the line.

(vii) Putting $n = 2$ in the result obtained in §2, we find that the distance of the centre of gravity of a uniform triangular lamina from a line lying in the plane of the lamina, the density at any point of the lamina varying as the pth power of the distance from the line, is the same as that of particles of masses proportional to $\left(\frac{\partial S_{3,p+1}}{\partial u_1}, \frac{\partial S_{3,p+1}}{\partial u_2}, \frac{\partial S_{3,p+1}}{\partial u_3}\right)$ placed at the vertices of the lamina.

(viii) It can be easily seen that distance of the centre of pressure of a plane triangular lamina from a line lying in the plane of the lamina, the density of the fluid at any point varying as the pth power of the distance from the line, is the same as the distance of centre of gravity of masses proportional to $\left(\frac{\partial S_{3,p+2}}{\partial u_1}, \frac{\partial S_{3,p+2}}{\partial u_2}, \frac{\partial S_{3,p+2}}{\partial u_3}\right)$ placed at the vertices of the lamina.

(ix) Putting $p = 0$ in (viii), we find that the distance of the centre of pressure of a uniform triangular lamina immersed in a homogeneous fluid, from the line of intersection of the effective surface and the plane of the lamina, is the same as the distance of the centre of gravity of masses proportional to $\left(\frac{\partial S_{3,2}}{\partial u_1}, \frac{\partial S_{3,2}}{\partial u_2}, \frac{\partial S_{3,2}}{\partial u_3}\right)$ placed

at the vertices of the lamina, where

$$\frac{\partial S_{3,2}}{\partial u_1}=2u_1+u_2+u_3, \frac{\partial S_{3,2}}{\partial u_2}=2u_2+u_3+u_1, \frac{\partial S_{3,2}}{\partial u_3}=2u_3+u_1+u_2.$$

(c) Putting $p=0$ in equations (2.12) - (2.15) we find that for a simplex imbedded in an E_n, [1]

$$\bar{z}=\frac{1}{n+1}\frac{S_{n+1,1}}{S_{n+1,0}}$$
$$=\frac{A_{n+1,1}}{A_{n+1,0}}$$

and

$$k^2=\frac{2}{(n+1)(n+2)}\frac{S_{n+1,2}}{S_{n+1,0}}$$
$$=\frac{A_{n+1,2}}{A_{n+1,0}}$$
$$=\frac{2}{(n+1)(n+2)}\{u_1{}^2+u_2{}^2+\ldots+u_{n+1}{}^2+u_1u_2+$$
$$+\ldots+u_pu_q+\ldots+u_nu_{n+1}\}.$$

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Note on a pair of combinatorial reciprocal formulas

By L. C. Hsu

1. Reciprocal formulas. In this note we shall consider a pair of reciprocal formulas as follows:

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k g(k), \tag{1}$$

$$g(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k), \tag{2}$$

where $f(0) = g(0)$ and $n = 0, 1, 2, 3, \ldots$.

If we call $f(n)$ as defined by (1) the combinatorial transform of $g(n)$, then the reciprocity of (1) and (2) simply means that *f and g are combinatorial transforms of each other.*

As is easily observed, the reciprocal formulas (1) and (2) are actually equivalent to the Newton interpolation formula and the formula for higher differences. In fact, by the formula for higher differences we deduce immediately from (2) that

$$g(n) = (-1)^n \Delta^n f(0), \qquad n = 0, 1, 2, \ldots . \tag{2'}$$

Consequently, upon substitution we see that (1) takes the form

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0), \tag{1'}$$

which is the well-known formula of interpolation.

The reciprocity of (1) and (2) can also be established quite directly. It is known that

$$\sum_{\nu=p}^{n} (-1)^\nu \binom{n}{\nu} \binom{\nu}{p} = \binom{n}{p} \sum_{\nu=p}^{n} (-1)^\nu \binom{n-p}{\nu-p}$$

$$= \begin{cases} 0 & (p < n), \\ (-1)^n & (p = n). \end{cases}$$

Thus if (1) is assumed, we observe that the right-hand side of (2) becomes

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\sum_{\nu=0}^{k}\binom{k}{\nu}(-1)^\nu g(\nu)\right)$$

$$=\sum_{\nu=0}^{n}(-1)^\nu g(\nu)\left(\sum_{k=\nu}^{n}\binom{n}{k}\binom{k}{\nu}(-1)^k\right)$$

$$=\sum_{\nu=0}^{n}(-1)^\nu g(\nu)\begin{Bmatrix}0 & (\nu<n)\\ (-1)^n & (\nu=n)\end{Bmatrix}$$

$$=(-1)^n g(n)\,(-1)^n=g(n).$$

This shows that (2) is implied by (1). Similarly, we see that (1) is also implied by (2).

More generally, we have the following pair of reciprocal formulas :

$$F(n_1,\ldots,n_s)=\sum_{0\leqslant k_i\leqslant n_i}\binom{n_1}{k_1}\cdots\binom{n_s}{k_s}(-1)^{k_1+\cdots+k_s}G(k_1,\ldots,k_s), \quad (3)$$

$$G(n_1,\ldots,n_s)=\sum_{0\leqslant k_i\leqslant n_i}\binom{n_1}{k_1}\cdots\binom{n_s}{k_s}(-1)^{k_1+\cdots+k_s}G(k_1,\ldots,k_s), \quad (4)$$

where $n_i=0,\ 1,\ 2,\ 3,\ldots\ (i=1,\ 2,\ldots,\ s)$ and $F(0,\ldots,\ 0)=G(0,\ldots,\ 0)$.

The proof is essentially the same as that of the case $s=1$, and may therefore be omitted.

2. Examples.

EXAMPLE 1. *For any positive integer n we have*

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k S_k=-\frac{1}{n},$$

where $S_1=1,\ S_k=1+\frac{1}{2}+\cdots+\frac{1}{k},\ (k=2,\ 3,\ 4,\ldots).$

PROOF. The identity states that $T_n=-1$ is the combinatorial transform of S_n. By reciprocity it suffices to consider the inverse transform

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k T_k=S_n.$$

But this is the familiar identity

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \dots + (-1)^{n-1}\frac{1}{n}\binom{n}{n}$$
$$= 1 + \frac{1}{2} + \dots + \frac{1}{n},$$

obtained very easily by considering both sides of the integral

$$\int_0^1 \frac{1-(1-x)^n}{x}\,dx = \int_0^1 \frac{1-x^n}{1-x}\,dx.$$

EXAMPLE 2. *Let n, m, s be any positive integers. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k+m}{S}(-1)^k = \begin{cases}(-1)^n\binom{m}{s-n}, & (m \geqslant s-n \geqslant 0)\\ 0, & (m < s-n \text{ or } s-n).\end{cases}$$

PROOF. The most familiar identity

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{s-k} = \binom{n+m}{s}$$

may be expressed as

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k) = g(n),$$

where

$$f(k) = (-1)^k\binom{m}{s-k},\ g(n) = \binom{n+m}{s}.$$

Thus by reciprocity we get what we want to prove.

EXAMPLE 3. *Let m distinct books be sent to n friends such that each friend must receive at least one. Then the number of ways of all possible distributions is given by*

$$A_n = \sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}k^m,\ (m \geqslant n).$$

PROOF. If the condition that every friend must get at least one book had been relaxed, then the number of ways of distributions is obviously n^m, because each book can be sent to any one of the

n friends. Now let A_k denote the number of all such possible ways that just k fixed friends receive books. What we want to seek for is the number A_n.

Since the total number of ways is n^m, we easily observe that

$$A_n + \binom{n}{1} A_{n-1} + \binom{n}{2} A_{n-2} + \cdots + \binom{n}{n-1} A_1 = n^m.$$

This may be written as

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k \left((-1)^k A_k \right) = B_n, \quad B_n = n^m.$$

Hence by the reciprocity we obtain

$$A_n = (-1)^n \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^m.$$

In particular, if $m = n$ then each friend just gets one book so that A_n is precisely the number of all permutations, i.e.

$$n! = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} k^n.$$

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CLASSROOM NOTES

A geometric solution of the cubic equation using a central conic

By T. Krishna Rao, *Vaniyambadi*

1. It is well known that it is impossible by means of the ruler and compasses alone to (1) trisect an angle (2) to duplicate the cube or (3) more generally to get the roots of the equation $z^3 + 3qz + r = 0$. There are however many methods known to effect the first two constructions by the use of curves other than the circle and the straight line. It is also known that the third construction can be reduced if $q < 0$ and $1 + \frac{r^3}{4q^3} > 0$ to the first construction and to the second by the Cardan's method if $1 + \frac{r^3}{4q^3} < 0$.

In the absence of any mention in the more easily accessible references, we give below the following methods of effecting the above constructions, by means of a given ellipse in case (1) and by means of some central conic in the other cases. In case (1) the crux of the method consists in the well-known fact that there are precisely three points ϕ on an ellipse, the circle of curvature at which passes through a given point θ. Also ϕ will be equal in magnitude to $\frac{\theta}{3} \bmod \frac{\pi}{3}$ and the method consists just in identifying this circle.

2. **A circle associated with a point on a central conic.**

2.1. We associate with any point P on a given central conic Γ the circle $\Gamma(P)$ passing through P and having its centre M at the mid-point of the segment joining centre O of Γ to the mid-point of the intercept KL made by the axes of Γ on the normal to it at P.

If Γ is given by the equation $\frac{x^2}{a} + \frac{y^2}{b} = 1$ and P is the point (x_0, y_0) on Γ, it is easily verified that the centre M of $\Gamma(P)$ has coordinates $\left(\frac{a-b}{4a}\, x_0, \frac{b-a}{4b}\, y_0\right)$ and $\Gamma\,(P)$ has the equation

$$x^2 + y^2 - \frac{a-b}{2a}\, x_0 x - \frac{b-a}{2b}\, y_0 y - \frac{a+b}{2} = 0.$$

We might mention the following property of $\Gamma(P)$:

2.2. $\Gamma(P)$ passes through the foot of the perpendicular from the centre of Γ to the tangent at P and also through the feet of the perpendiculars from the foci on to the reflection of this tangent about the transverse axis.

The analytic verification is straightforward and is omitted. A more important property of $\Gamma\,(P)$ is given in :

2.3. Theorem. *There are precisely three points on a central conic* Γ, *the circles of curvature at which pass through a given point* P *on* Γ. *These points are the intersections besides* P *of* Γ *and* $\Gamma\,(P)$.

Proof. Using the fact that opposite pairs of common chords of a circle and a central conic are equally inclined to the axes (and conversely) we see that the circle of curvature at $P_1, (x_1, y_1)$ on Γ passes through $P\ (x_0, y_0)$ if and only if PP_1 and the tangent at P_1 are equally inclined to the axis, i.e. have their $m-s$ equal to the negative of each other. The m of P_1P is then equal to $\frac{bx}{ay}$ and it is easily seen that ;

$$x_0 = (4x^3{}_1 - 3ax_1)/a;\ y_0 = (4y^3{}_1 - 3by_1)/b.$$

Thus given (x_0, y_0), P_1 are seen to be the points whose x-and y-coordinates are given respectively as the roots of the equations

$$4x^3 - 3ax - ax_0 = 0;\ 4y^3 - 3by - by_0 = 0.$$

It follows that there are just three points. These points then satisfy the following equations ;

$$4x^3 - 3ax - ax_0 = 0; \tag{1}$$

$$4y^3 - 3by - by_0 = 0; \tag{2}$$

$$\frac{x^2}{a} + \frac{y^2}{b} = 1. \tag{3}$$

From (1) and (3) we get $4x\left(\frac{1}{4} - \frac{y^2}{b}\right) = x_0$; similarly, $4y\left(\frac{1}{4} - \frac{x^2}{a}\right) = y_0$. Substitution for (x_0, y_0) then shows easily that

$$x^2 + y^2 - \frac{a-b}{2a}x_0 x - \frac{b-a}{2b}y_0 y - \frac{a+b}{2} = 0,$$

i.e. the circle through these points is precisely $\Gamma(P)$.

It is to be remarked that this result is a well-known consequence (in the case of the ellipse) of the fact that the points θ_1, θ_2, θ_3, θ_4 are concyclic, if and only if, $\Sigma\,\theta_1 = 2n\pi$. Thus the circle of curvature at ϕ passes through θ if and only if

$$\phi = -\frac{\theta}{3} \text{ or } -\frac{\theta + 2\pi}{3}, \text{ or } -\frac{\theta + 4\pi}{3}.$$

3. Geometric construction for the trisection of an angle by the use of ruler and compasses and a given ellipse.

Let P be the point with eccentric angle θ on Γ. Let P_1, P_2, P_3 be the remaining intersections with Γ of $\Gamma(P)$. Then the eccentric angles of each of P_1, P_2, P_3 is equal in magnitude to $\frac{\theta}{3} \bmod \frac{\pi}{3}$.

The proof follows from the remark at the end of the preceding paragraph and from Theorem 2.3.

4. Geometric solution of the cubic equation.

We may take $z^3 + 3qz + r = 0$ as the given equation. We notice that the roots of this equation are the x-coordinates of the other intersections of Γ and $\Gamma(P)$, i.e. are the same as the roots of the equation (1) in §2 : $4x^3 - 3ax - ax_0 = 0$ provided only

$$a = -4q; \; x_0 = -\frac{4r}{a} = \frac{r}{q}.$$

We are thus led to a geometric construction to the roots of the given equation provided only we can choose the conic $\frac{x^2}{a} + \frac{y^2}{b} = 1$ and the point (x_0, y_0) on it so as to be real and so as to satisfy the equation (3).

Now $y_0 = \left\{ b\left(1 - \frac{x^2_0}{a}\right)\right\}^{1/2} = \left\{ b\left(1 + \frac{r^2}{4q^3}\right)\right\}^{1/2}$ is real, if and only if, $b\left(1 + \frac{r^2}{4q^3}\right) > 0$ which can always be adjusted by proper choice of the sign of b. But then the conic may fail to be real, if and only if, both a and b are negative, i.e. when $q > 0$ and $1 + \frac{r^2}{4q^3} < 0$ which cannot be. Thus the method will always work and we can state :

4.1. Rule : To solve the equation $z^3 + 3qz + r = 0$ geometrically, take the conic $\Gamma : \frac{x^2}{a} + \frac{y^2}{b} = 1$, where $a = -4q$ and b is any convenient number subject only to the condition that it is to have the same sign as $1 + \frac{r^2}{4q^3}$ except that when Γ is an ellipse b is also $< a$. Let P be the point on Γ whose x-coordinate is $\frac{r}{q}$. Then the roots of the given equation will be the x-coordinates of the intersections of Γ with the circle $\Gamma(P)$.

BOOK REVIEWS

Analytical plane geometry. By Jagjit Singh, Lahore Bookshop, Ludhiana (1954), Second Edition, 360 pp., Rs. 7/8.

This is a textbook on analytical geometry dealing with the usual properties of the straight line, the circle and the conic, 'intended for B.A. students of *our* universities'. It claims to possess some new features which the reviewer has in the main been unable to detect. One obvious feature, however, is its length ; without the tiresome repetitions with which the book is replete the book could easily have been reduced to two-thirds its present length without prejudice either to clarity or content. With all the verbiage expended in every context, some of the essential points are never stressed. As far as the reviewer could trace, it has nowhere been mentioned that a point corresponds to an *ordered* pair of numbers. Cases where some of the coefficients of an equation may be vanishing have not been dealt with with sufficient care. Loose statements abound, and the explanations offered, in several cases, hinder rather than help the understanding of the text. (*Cf.* pp. 1, 22, 146).

The printing is primitive ; the proof reading has been careless, and the errata listed at the beginning do by no means tell the whole story. A typographical feature which cannot escape attention is the resort to italics on the slightest provocation. One would hardly consider the book a welcome addition to the host of textbooks on the subject.

K. BALAGANGADHARAN

A concise history of mathematics. By Dirk J. Struik, G. Bell and Sons Ltd., xix + 299 pp., 14 sh.

To condense a history of mathematics into a book of about three hundred pages is certainly a formidable task requiring the best qualifications as far as mathematics as well as history are concerned. Professor Struik is a scholarly mathematician ; but there is little

doubt that he is not a full fledged historian. His *Concise History of Mathematics* shows extensive reading, the sincere resolve to offer us a vivid picture of the development of mathematics since times " as far removed as the Old Stone Age, the Paleolithicum " down to the beginning of the XX century. Unfortunately, it is one thing to avoid the monotony of a dry enumeration of facts and names, another to do it by resorting to the objectionable method of writing as a man " with his likes and dislikes, his knowledge and his ignorance ". Likes and dislikes may be condoned from time to time in the work of a learned and outstanding historian. They are deplorable in the work of an amateur.

One is really amazed at the facility with which likes and dislikes goad Prof. Struik to speak of the old stone age as if the most detailed records were still extant. One smiles—or is irritated, according to the disposition of the moment—when one is taught the exact relations between economic and social conditions, and the growth of mathematics in particular and knowledge in general, as if this was a matter as clear as the theory of the quadratic equation. Sometimes one regrets much the lack of foot-notes to explain, or justify, sweeping statements which are either ambiguous or totally puzzling. What is to be understood, for instance, by the " democratic element " which in the V century " was the driving force behind the economic and military expansion of Athens " ? As a matter of fact, not only a foot-note but an appendix would have been most welcome to let us know the meaning attached by the author to the magic epithet " democratic ". This would have made it easier to understand why " despite his reactionary ideals " Cauchy has been able to produce a work of fundamental importance ; or why " the mystical period (to use a term suggested by Karl Marx) ", which is that of the foundation of the Calculus, " provoked a mysticism which occasionally went far beyond that of the founding fathers ". Indeed the student of various forms of mysticism knows full well that in most cases this is hardly avoidable. I am afraid this is rather pitiless criticism. Yet I think that mathematicians will easily find in the first six chapters of this book valuable informa-

tion, along with a rich bibliography—at any rate much material for fruitful reflections. Unfortunately the last two chapters, resulting from an excessive process of condensation, are too sketchy and make no profitable reading.

The book contains in addition to an extensive literature very nice illustrations.

C. RACINE

Electromagnetic theory. By V. C. A. Ferraro, The Athlone Press, University of London, 1954, 555 pp., 42 sh.

The book is divided into three parts. The first part deals with electrostatics, magneto-statics and steady currents; the second part gives special methods for solving the boundary-value potential problems ; and the last part contains Maxwell's equation and a brief account of electro-magnetic waves, radiation and motion of an electric charge.

The gaussian unrationalised system of units has been used throughout the book. Use of vector methods has made the treatment simple and elegant.

The author has adopted a classical approach following the historical order of development of the subject. No doubt, such an approach is convenient in developing the mathematical theory, yet an approach based on the ideas of atomic physics would have been realistic and more convincing.

Following the classical treatment the author has developed the theory of magnetism through the notion of magnetic dipole. His proof of the equivalence of a current flowing in a closed circuit and a magnetic shell is novel and more direct than the usual proofs.

The theory has been amply illustrated with solved examples and at the end of each chapter a set of well-selected problems has been given. The problems have been mostly drawn from the examination

papers of Cambridge and London universities. Most of the examples in the first part are, however, similar to those given in other standard text books.

The treatment is rigorous, yet simplified. The book adequately meets the requirements of the students reading for the Master's degree of Indian universities.

The book fills the long-felt need of a comprehensive textbook treating the subject with the aid of vector methods, and as such, in the opinion of the reviewer, it will be widely welcomed.

P. L. BHATNAGAR

Sampling theory of surveys with applications. By Pandurang V. Sukhatme. The Indian Society of Agricultural Statistics, New Delhi, India, and The Iowa State College Press, Ames, Iowa, U.S.A. 1954, xxix + 491 pp.

1. The growing demand of sampling methods in surveys has created the continuing need of up-to-date and systematic treatises on the theory and practice of this subject. Indeed the publication of such books has in no small way contributed towards the extended use of sampling methods in various fields of immense practical importance. Although reputed workers have already brought out very useful treatises on this subject a book from the pen of Dr. Sukhatme is certainly welcome because he has, specially in relation to agricultural statistics, the necessary back-ground information about conditions in the so-called under-developed countries as typified by India, and as such he is likely to be in a position to draw attention and offer solutions to problems peculiar to those countries.

2. The book is divided into ten chapters. The first two deal with basic ideas and basic theories ; the author next passes on to a fairly comprehensive account of stratified sampling. He then devotes two chapters to the ratio and regression methods of estimation.

The theory of cluster sampling is taken up next. The several pages of the subsequent two chapters contain an account of multi-stage sampling and these are followed by one on systematic sampling. Dr. Sukhatme then completes his book with some discussion on the important topic of non-sampling errors.

3. The first chapter on basic ideas, which is of fundamental importance, perhaps could have been written with much greater care. Generally speaking, the reader who is already more or less familiar with the subject matter may not find it difficult to follow exactly what the author desires to convey, but to a beginner this may not be so. The treatment is sometimes very terse, jumps abruptly from one idea to another and the connecting link is not always obvious; moreover, the emphasis is not always at the proper place. At times it uses concepts not previously explained.

4. The most striking feature of the book is its very full algebraic treatment of the subject matter. The author has taken special care to write down in great detail all steps involved in the mathematical derivation of the results. In some cases he has given more than one proof. He has consistently treated the cases of selection with equal and unequal probabilities in separate sections. He has also considered it worthwhile to repeat the whole process step by step even in cases which are not essentially different as regards the method of derivation. Such a treatment no doubt involves a good deal of space but the reviewer thinks that this approach will be liked by a large number of students preparing for the university examinations. They will also find it a great comfort to go through the various steps of the mathematical portions because the publishers have not tried, as is usually done, to compress a large volume of mathematical matter in a relatively small space.

5. The treatment of this subject does not usually involve very advanced mathematics. There are a few cases of the problem of maxima and minima arising in the question of optimum allocation of the sample which are perhaps not quite straightforward, and here some explanatory details would have been very welcome to the

reader. Thus, for example, in the problem of allocation in a three stage sampling scheme (p. 311) one encounters the problem of determination of positive numbers (integers) m and p which minimize

$$c_2 Pm + c_1 B/m + c_3 Bp + c_2 W/p + c_3 Pmp + c_1 W/mp,$$

the c's and W being positive. In considering the case $P \leqslant 0$ and $B > 0$ it is simply stated without any discussion that to minimize the above expression m must assume the *maximum attainable value* (reviewer's italics) say $\boldsymbol{m}$, and then if $c_3 B + c_3 \boldsymbol{m} P$ turns out to be non-positive, p is to be given the maximum attainable value. It is not explained what is meant by maximum attainable value and how $\boldsymbol{m}$, for instance, is to be determined. Again in some of the discussions on optimal allocation, the problem is not tackled in its full generality, for example, the restriction that the sample size for any stratum cannot exceed the number of sampling units in that stratum is not taken into account. The reviewer is of the opinion that theoretical discussions should be preferably in full generality ; but if certain cases are omitted from consideration then this should be clearly stated.

6. In his discussions on regression estimates (and also some portions of ratio estimates) the author assumes a very restricted type of finite population: N_i units each of value x_i $(i = 1, 2,\ldots,k)$ and the mean value $\bar{y}_{i\cdot}$ of y (the population total or mean of which is to be estimated) for a given x_i is exactly equal to $\alpha + \beta x_i$, α and β being independent of x_i. Also the method of sampling is at times only restricted to sampling a fixed number of units from each of the k classes. How far this approach is in conformity with the regression and ratio methods as ordinarily used is not very clear.

7. The author's attempt to develop a unified system of notation for use in mathematical discussions is highly commendable. Some confusion may perhaps arise at times, e.g., the two $\bar{y}_{n\cdot}$'s in the relation (p. 288)

$$y_{n\cdot} = \frac{1}{n} \sum^{n} \bar{y}_{i\cdot}$$

have different meanings. The left-hand $\bar{y}_{n\cdot}$ denotes the simple mean of n first-stage unit means, whereas $\bar{y}_{n\cdot}$ in the sense of the right-hand side may be taken as the mean per second-stage unit in the n-th first-stage unit in the population. Use of $\bar{\bar{y}}_{n\cdot}$ in place of the left-hand $\bar{y}_{n\cdot}$ may remove this slight confusion.

8. In writing up a review for a mathematical journal the reviewer cannot help referring to a type of argument which unfortunately has entered the statistical literature and a copious use of it has been made in this book. This is fully illustrated by the following extract (p. 37):

$$\text{“}\, s = S\left(1+\frac{\epsilon}{S^2}\right)^{1/2}.$$

Since ϵ will be small as compared with S^2 with a probability approaching 1 as n becomes large, we may expand the right hand side as a series

$$s = S\left\{1+\frac{1}{2}\cdot\frac{\epsilon}{S^2}+\frac{\frac{1}{2}(-\frac{1}{2})}{1.\,2}\left(\frac{\epsilon}{S^2}\right)^2+\dots\right\}.$$

Taking expectation of both sides we obtain...”. Obviously this is not a rigorous argument, and that this is so should at least be clearly stated, if nothing better can be found. The mathematician will also find it difficult to accept without reservation some of the results "proved" in the book in connection with a random variable which assumes with known probability an unknown or indeterminate value (e. g., in 2*a*. 19 when $n_1 = 0$).

9. Incidentally the reviewer feels that in view of the lack of rigour in the above type of argument it is perhaps not quite worthwhile to make any additional effort covering some 10 pages (including appendix) on what is called the second approximation to the expected value and to the variance of the ratio estimates. Another instance of somewhat of an over-devotion to certain points is the discussion on expected values of symmetric functions which

together with the necessary partitional notation and tables account for some 13 pages and all these have been used only to obtain $V(s^2)$, the sampling variance of s^2. This stands in great contrast to the total omission of any discussion on the practical importance of this result.

10. Some illustrative examples which will be quite useful to students are given. In certain cases the student may find some difficulty because the illustrations may not strictly be examples of the theory given earlier. A fair proportion of the examples make use of the data of (complete) census in comparing the relative efficiencies of different methods. There are a few examples of the use of an actual sampling enquiry in assessing the value of alternative sampling (including estimation) methods but some more would have been welcome. In connection with Table 6.10, it is not clear why in spite of the fact that an individual farm gives a smaller standard error compared to S_4 for each and every one of the 18 items under investigation the latter is considered to be the optimum sampling unit.

11. The abrupt manner in which certain conclusions are drawn may be somewhat bewildering to the reader. In example 6.5 it is shown that, in estimating the total area under wheat for Hapur Sub-division, a sample of n circles (a compact cluster of villages under a *Patwari* is called a circle) gives rise to a larger sampling variance for certain methods of sampling and estimation than the " equivalent " number of villages (i.e. $n\bar{M}$ villages, $\bar{M}$ being the average number of villages per circle). It is then concluded, " The village as a unit of sampling is seen to be far superior to the circle, and since the village is also known to be administratively convenient, it is preferred to the use of the circle as the unit of sampling in most agricultural surveys in India. " The reader may wonder, if such a decision could be made without any reference to cost considerations then what stands in the way of selecting an " equivalent " number of fields (survey numbers), which would give rise to a sampling variance much smaller than that given by the

village. Moreover how the circle which is an "administrative" unit is administratively less convenient than the village, as perhaps is implied, is not explained. The author no doubt had the question of costs in addition to other factors in his mind in the choice of the village as the sampling unit, but this point is not clearly brought out in the above passage.

12. The author has very rightly emphasised that mere biasedness in an estimate is not a sufficient ground to reject it in favour of an unbiased estimate, because in spite of the bias, the mean square error (MSE) of the former may be less than that of the latter. But the great danger of using a biased estimate, in the opinion of the reviewer, lies in the fact that the great simplicity in computation, which is usually associated with the biased estimates ordinarily considered, leads one to the wishful thinking that the bias is negligible and as a natural consequence one is too readily tempted to accept this convenient assumption on inadequate experience or experience in an entirely different field. One may even be dogmatic in practice in accepting as an almost universal truth, that a particular biased estimate is more efficient than a given unbiased one. The author seems to lean towards this attitude when at the very beginning of his discussion on relative efficiencies of different estimates (p.272) he summararily rejects the unbiased estimate $\bar{y}'_{n\cdot}$, by simply saying that it will generally be less efficient than the others. But strangely enough he himself gives an example (Ex. 6.5) immediately afterwards on the important problem of estimation of crop acreage in which the unbiased estimate $\bar{y}'_{n\cdot}$, has a much lower MSE than that of the biased estimate $\bar{y}'_{n\cdot}$, and what is more, even the sampling variance of the latter (obtained by neglecting the bias component) is greater than that of the unbiased estimate. In giving this example probably the author's intention is to focus attention to the desirability of taking into consideration the bias also in assessing the accuracy of the estimate. Fortunately in this example there is no difficulty in finding the exact value of the bias as the data for the entire population is being used. But how one is to proceed, in the more common situation, when only

the results for a sample and not those of the entire population are available, is not explained.

13. The question of negligibility or otherwise of the bias is rather difficult to decide in several situations. In the extremely important survey conducted by the Indian Council of Agricultural Research (ICAR) for the determination of crop yields in a large part of India, a biased estimate, which is computationally very convenient, is used. The author has taken pains in his attempts to demonstrate the negligibility of the bias. But his argument is not convincing and the whole matter appears to be rather puzzling. The reviewer has not been able to reconcile the somewhat divergent statements contained in the book with those in the paper by Sukhatme and Panse (1951) and the Report of the Wheat Survey in the Punjab 1943-44, a reference to both of which has been made in the book. The book discusses the yield surveys under two stage sampling (p.321) whereas in the paper it is a three stage one; there are also differences in other respects which need not be stated here. The estimates in the book (Table 7.4) which are shown as $\bar{y}_s$ and $\bar{y}'_s$ are at the same time claimed in the paper to be obtained by different estimators, ($\bar{x}_{nmp}$ and $\bar{x}''_{nmp}$ according to the notation of the paper). Although the estimates shown under $\bar{y}'_s$ are claimed to be unbiased in the book, the same figures shown in the paper are the estimates $\bar{x}''_{nmp}$, which the paper itself points out to be biased. The Punjab Report adds to this confusion. The estimator $\bar{x}''_{nmp}$ takes into account inequalities in field as well as the village crop areas ; but in obtaining the actual estimates the Report states that only corrections for inequality in field areas have been made, and the other correction (perhaps more important) for village crop area inequalities has not been made, so that what is claimed in turn as $\bar{x}''_{nmp}$ in the paper is, as far as the reviewer can make out, not strictly so. (It is claimed in the Report without any supporting figures that the source of bias arising from inequalities in village crop areas is wholly negligible. It is not clear why estimates incorporating both the corrections are not reproduced in place of the partially corrected, and therefore inferior, estimates).

14. Even leaving apart the above points the reasons put forward by the author in support of his thesis, that the bias is negligible, may not be acceptable without further explanation. An estimate of the bias is given by the difference and it is stated that the (district) estimates of the bias are known to have standard errors of the order of 6 to 8%. The S.E.'s are presumably expressed as percentages of the yield per acre. (The possibility of the S.E.'s being expressed as percentages of the biases is ruled out because the claim on page 322 that the differences or estimated biases are negligible compared to their S.E.'s then would not be substantiated). An examination of Table 7.4 shows that in all the 11 districts the estimated bias is usually very much less and never greater than 5% of the yield per acre. This is to be hardly expected on the hypothesis that the bias in the district estimates $\bar{y}_s$ is zero. Leaving aside this point, it is true that for each district the estimated bias is small, at least not large, compared to its S.E., but when this is put forward as a ground for considering the bias to be negligible then one may rightly feel that bias of any magnitude can be shown to be negligible if only an inadequate sample is taken, so that the S.E. of the estimated bias may be comparatively high. Moreover, in view of the largeness of the S.E. (of the bias) in relation to the bias, it is no wonder that the *estimated* bias changes sign from district to district, and there is no justification to conclude, contrary to what is apparently implied in the book, that the *actual* direction of the bias also changes from one district to another. The real question is whether one is prepared to neglect possibly large biases which are apt to be overlooked when one uses such a large yard-stick (a S.E. 6-8% of the yield). In any case it will be completely deceptive to judge the accuracy of a biased estimate by merely its sampling variance, as distinct from MSE, unless the sampling variance is large compared to this yard stick. Table 7.5 showing, for example, the percentage S.E.'s of mean yield estimates $\bar{y}_s$ and $\bar{y}'_s$ as 3.7 and 14.0 respectively for wheat (UP. 1947-48) should be judged with the above point in mind. It may be pointed out in this connection that the S.E. of the difference

$\bar{y}_s - \bar{y}'_s$, as can be easily shown, is greater than the difference between the S.E.'s of $\bar{y}_s$ and $\bar{y}'_s$; and as such the length of the yard-stick in the above example is more than 10.3% of the mean yield which is very much large compared to the percentage S.E. of 3.7 of the biased estimate $\bar{y}_s$.

15. In making the above comments the reviewer's intention is to point out the desirability of having a fuller discussion on these difficulties which the reader may face in following the argument. It is possible and perhaps very likely, because the question of bias in yield estimates must have received very serious attention, that the author is really in a position to dispel all these doubts and the reviewer wishes that, for the benefit of the readers, these points will receive sufficient attention in subsequent editions of the book. The yield survey is of extreme importance and because of this the reviewer hopes that the detailed comments on this particular item will not appear to be disproportionate.

16. The book is entitled " Sampling Theory of Surveys with Applications " and one would therefore expect a substantial portion of the book devoted not only to questions of theoretical interest but also those on practical difficulties in planning, execution, tabulation, and analysis of a sampling enquiry. Unfortunately these latter problems are discussed only incidentally and therefore very briefly. Sometimes such remarks are so brief that one is left wondering why the author is saying so. Thus (p.84) he states without explanation, " The selection of sampling units, the location and enumeration of the selected units and the distribution and supervision of field work are all simplified in stratified sampling. " The next sentence, " Of course, stratified sampling presupposes the knowledge of the strata sizes,. . . " is not much of an explanation. Clarification of such passages would be of great value to planners of sample surveys.

17. There are considerable gaps in several directions between theory and practice, and some discussion on these would have been

desirable specially when these questions naturally arise from the examples given by the author. Thus in Ex. 1.2 we are told that the field selection procedure adopted by ICAR in 1951 did not give equal chance to every paddy growing field (Survey number), presumably, contrary to what was actually desired. In practice perhaps this was the best course possible, but it is important to assess the effect of this procedure on the final yield estimate, and some discussion would appear to be very desirable specially when so much space has been devoted throughout the book to the question of yield estimates.

18. In another question of practical importance, viz. the role of interpenetrating sub-samples in the control of field work the author has definite views. One must draw his own conclusions for his own guidance only after giving the method a fair trial. In the reviewer's experience, even leaving apart other advantages of such samples, the very knowledge of the existence of interpenetrating sub-samples works as a deterrant to slipshod work. The author has taken pains to show that "Replicated (interpenetrating) samples are no alternative to supervisory check"; the reviewer only wonders whether in any survey actually conducted, such samples have really replaced supervisory check or these have been used as an aid or supplement to the latter. It would be interesting to have the views of the author as to other uses of such samples, for example, in controlling large scale tabulation work.

19. It is not sufficient for an author to claim the superiority of the design of ICAR with which he was associated over that used by another without giving adequate reasons. The method followed by ICAR in the estimation of crop acreages in Orissa is claimed to be superior to that followed in Bengal by the Indian Statistical Institute (ISI) as reported in the papers by Mahalanobis to which a reference has been made in the book. This claim is made without any reference to relative sampling variance and costs and field agencies. It is only pointed out that the square grids used by ISI are artificial and therefore inconvenient for field work. If the author

had only studied the method carefully he would have seen that from the field point of view the clusters to be surveyed were as much "natural" as those used in Orissa. The author is perhaps under the misconception that these investigators had to report only on the part of the field which fell within the grid ; this however is not so, the investigator had to report on the whole field irrespective of whether it was partly or wholly covered by the grid. Part inclusion of a field was taken into account at the estimation stage. A second point advanced in favour of the Orissa method is that to increase the efficiency it (presumably) left out the single survey number marked as uncultivated in the village map, whereas in the Bengal method it was not. The advisability of assuming that the survey number, which happened to be uncultivated perhaps some decades ago at the time of cadastrial survey, is still wholly uncultivated is highly questionable. Perhaps some further details may be found in the Orissa Report to which the reader is asked to refer ; unfortunately the Report is unpublished. Incidentally the book contains reference to a large number of unpublished reports and it is highly desirable that these reports be published for the benefit of all concerned.

20. Coming now to an entirely different point the reviewer would like to draw the attention of the author that in naming a certain law after Jessen he has overlooked that Mahalanobis arrived at the result earlier, a fact which he has himself noted (p. 255) in his book. Attention to two very emphatic but inconsistent statements may also be drawn. These are, "Simple random sampling is by far the most common method of sampling in surveys" (p.20) and "By far the most common design in surveys is stratified multistage sampling" (p.349).

21. The book is the outgrowth of the author's lecture notes and his problem was to use these to produce an integrated whole in the form of a book. Dr. Sukhatme is to be congratulated that even with the heavy responsibility which his present assignment as chief of the Statistics Branch of the Economics Division of the

Food and Agricultural Organization of the United Nations necessarily involves he has carried out this work creditably, even though there is some scope for improvement. The comments made earlier on a few short-comings are made with the hope that these will serve to point out the directions in which such improvement is possible; but these comments are never meant to cloud an otherwise valuable contribution in the field of sample surveys.

22. The Bangalore Press who were responsible for the printing have done an excellent job. The choice of a map of a typical Indian village for the cover flap has been very aptly made.

D. B. Lahiri

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MADRAS

NEWS AND NOTICES

The Society has received a publication grant of Rs. 1,000/- from The Tata Institute of Fundamental Research for the year 1954-55.

The Government of India have reconstituted the National Committee for Mathematics in India as follows: Dr. H. J. Bhabha (*Chairman*), Prof. K. Chandrasekharan (*Convener and Secretary*), Prof. Ram Behari, Prof. S. N. Bose, Prof. S. Minakshisundaram, Prof. N. R. Sen, and Prof. T. Vijayaraghavan.

Professor Hans Rademacher of the University of Pennsylvania, and Professor Hans Maass of the University of Heidelberg will be visiting Professors in the Tata Institute of Fundamental Research for five months beginning November 1, 1954.

International Congress of Mathematicians 1954

The International Congress of Mathematicians (I.C.M.) meets once in four years. The first session after the war was held at Cambridge, Massachusetts, U.S.A. The session this year was held at Amsterdam, Holland, from September 2 to September 9. It was the largest Congress in the history of mathematics with 1729 regular members and several hundreds of associate members. The Congress was a great success from every point of view.

Over 60 outstanding experts accepted the Organizing Committee's invitations to give a survey of the recent developments in mathematics. Nearly 500 other mathematicians presented short papers. The scientific as well as the social programme was organized to near perfection and punctuality was the key-note. The Congress was held under the auspices of the Dutch Mathematical Society with the patronage of H. R. H. the Prince of the Netherlands. Professor J. A. Schouten was the President of the Congress and Professor J. F. Koksma the Secretary. The scientific programme was divided into seven sections: Algebra and Theory

of Numbers, Analysis, Geometry and Topology, Probability and Statistics, Mathematical Physics and Applied Mathematics, Logic and Foundations, Philosophy, History and Education. In each section there were invited addresses on the one hand and communicated papers on the other. In addition to these sections, there were three symposia (*a*) Stochastic Processes, (*b*) Algebraic Geometry and (*c*) Mathematical interpretation of formal systems, organized by the Wiskundig Genootschap at Amsterdam, in conjunction with the Congress, with financial assistance from UNESCO.

Inauguration: The Opening Session was held in the Amsterdam Concert Hall at 10.30 a.m. on September 2, with Professor Schouten in the Chair. After an official welcome, the session was addressed by Professor Hermann Weyl who was the Chairman of the Committee for the award of the Fields Medals. These medals are awarded every four years for outstanding young mathematicians who have made particularly significant contributions. The two recipients this year were Professor K. Kodaira of Japan and Professor J. P. Serre of France. After the presentation of the medals, the Secretary of the International Mathematical Union, Professor E. Bompiani, announced the names of the new Officers and Members of the Executive Committee of the I.M.U. (cf. Report of the Second General Assembly of the I.M.U.) Later the same evening an official reception by the Government of the Netherlands and the Municipality of Amsterdam was held in the Rijks Museum. A small number of selected invitees were presented to the Minister of Education of the Netherlands.

In the scientific sessions that followed, from September 3—September 9, the following mathematicians delivered Invited Addresses :

One-hour addresses.

P. A. Alexandrov (Russia), K. Borsuk (Warsaw), R. Brauer (Cambridge Mass.), J. Dieudonne (Ann Arbor, Mich.), S. Goldstein

(Haifa), Harish-Chandra (New York), B. Jessen (Copenhagen), A. Kolmogorov (Russia), A. Lichnerovicz (Paris), J. V. Neumann (Princeton, N. J.), J. Neyman (Berkeley), B. Segre (Rome), E. Stiefel (Zurich), A. Tarski (Berkeley), E. C. Titchmarsh (Oxford), K. Yosida (Osaka), A. Weil (Chicago).

Half-hour addresses.

Section I Algebra and Theory of Numbers: H. Davenport (London), P. Erdös (Los Angeles), E. Hlawka (Vienna), N. Jacobson (New Haven), H. Maass (Heidelberg), A. Neron (France), D. G. Northcott (Great Britain).

Section II Analysis : H. Behnke (Münster), F. Bureau (Liege), M. L. Cartwright (Cambridge, Eng.), L. Cesari (U.S.A.), K. Chandrasekharan (Bombay), A. Erdelyi (Pasadena), W. K. Hayman (Exeter), E. Hille (New Haven), P. J. Myrberg(Finland), C. Pauc (Nantes), Wazewski (Poland), A. Zygmund (Chicago).

Section III Geometry and Topology : G. Ancochea (Madrid) W. L. Chow (Baltimore), H. S. M. Coxeter (Toronto), B. Eckmann (Switzerland), H. Freudenthal (Utrecht), D. Montgomery (Princeton) H. S. Ruse (Leeds), J. P. Serre (Paris).

Section IV Probability and Statistics : J. L. Doob (U.S.A.), R. Fortet (France), M. G. Kendall (London).

Section V Mathematical Physics and Applied Mathematics : L. Collatz (Hamburg), G. Fichera (Triest), M. R. Hestenes (Los Angeles), J. Kampe de Feriet (Lille), J. J. Stoker (New York).

Section VI Logic and Foundations : P. Lorenzen (Bonn), A. Mostowski (Warsaw), J. B. Rosser (Ithaca).

Section VII Philosophy, History and Education : C. T. Daltry (England), J. E. Hofmann (Bayern), K. Piene (Oslo).

The addresses delivered will be printed as part of the Proceedings of the Congress.

There was an official Banquet on the evening of September 8 in the Winter Garden of the Grand Hotel Krasnapolsky. Professor

O. Bottema presided at the dinner. At the Banquet speeches were delivered, on invitation, by the Burgomaster of Amsterdam, Professor M. H. Stone (U.S.A.), Professor F. Severi (Italy), Professor P. A. Alexandrov (Russia), Professor A. Denjoy (France) Professor K. Chandrasekharan (India) and Professor W. V. D. Hodge (U. K.). All the speakers expressed their sincere appreciation of the excellent arrangements that had been made for the Congress and expressed the thanks of their respective countries to the organizers. The important difference between this year's Congress and the one held in America in 1950 consisted in the fact that many mathematically advanced countries like Russia, Hungary, Poland and Czechoslovakia which had not been represented in 1950 were fully represented at the present session. There was still a major country (China) which was not represented, and it was hoped that at the next Congress this gap would be filled. The concluding session of the Congress was held on September 9 in the Amsterdam Concert Hall. There was an address by Professor H. Hopf, the President-elect of the I. M. U. and it was announced that a Joint Committee of the I.M.U. and I.C.M. had recommended the acceptance of an invitation from the Royal Society of London, the Royal Society of Edinburgh, and from the City of Edinburgh, to hold the next Congress in August 1958 in Edinburgh. This recommendation was approved by acclamation.

INTERNATIONAL MATHEMATICAL UNION

Record of the Second General Assembly held on August 31—*September* 1, 1954 *at the Hague* (*Binnenhof*).

AGENDA OF THE MEETING

1. Action on seating Delegates.
2. Action on questions concerning membership.
3. *Appointment by the President of Working Committees.*
4. Resolution to form Nominating Committee—Implementation of same.
5. Reception of Reports.
6. Action on News Bulletin.
7. Budget for 1955-58. Discussion of contributing membership.
8. Proposals regarding symposia.
9. Work of the Executive Committee and Commissions—Addition and elimination of Commissions.
10. Discussion of relations with ICSU and other Unions.
11. Topics suggested by the National Adhering Organizations.
12. Elections.
13. Resolutions approving Reports.
14. Resolution on Congress—Next meeting of General Assembly.
15. Courtesy resolutions.

The initial session was opened at 9-30 by President M. H. Stone in the Binnenhof; later served as Chairmen of the various sessions Professors D. H. Kloosterman, A. Denjoy, S. Iyanaga, H. Kneser, B. Jessen, W. V. D. Hodge.

1. Action on seating Delegates.

The following persons were present at the meeting as Delegates or invited participants.

A. Delegates.

Argentina	Prof. A. Gonzales Dominguez
Australia	Prof. K. E. Bullen
Austria	Prof. N. Hofreiter, W. Gröbner

BELGIUM	Prof. L. Godeaux, Fl. Bureau
CANADA	Prof. G. de B. Robinson, R. L. Jeffery, H. S. M. Coxeter (alternate)
DENMARK	Prof. N. E. Norlund, W. Fenchel
FINLAND	Prof. P. J. Myrberg
FRANCE	Prof. A. Denjoy, J. Leray; A. Chatelet, Ch. Ehresmann, Ch. Pisot (alternates)
GERMANY	Prof. H. Kneser, K. Strubecker
GREAT BRITAIN	Prof. W. V. D. Hodge, M. H. Newman, E. T. Copson, H. Davenport, M. R. Cartwright
GREECE	Prof. C. Papaeoannu
ICELAND	Prof. L. Asgeirsson
INDIA	Prof. K. Chandrasekharan
ISRAEL	Prof. B. A. Amira, A. Fraenkel, M. Fekete
ITALY	Prof. F. Severi, G. Sansone, A. Signorini, A Terracini; E. Bompiani (alternate)
JAPAN	Prof. S. Iyanaga, K. Yosida
MALAYA-SINGAPORE	Prof. J. C. Cooke
MEXICO	Prof. G. Torres
NETHERLANDS	Prof. H. D. Kloosterman, O. Bottema
NORWAY	Prof. Th. A. Skolem
PORTUGAL	Prof. V. Gonzalves
SPAIN	Prof. J. Rey Pastor, R. San Juan Llosa
SWEDEN	Prof. A. Pleijel
SWITZERLAND	Prof. J. J. Burckhardt, E. Stiefel
UNITED STATES OF AMERICA	Prof. A. A. Albert, E. Hille, S. Mac Lane, E. J. McShane, M H. Stone
YUGOSLAVIA	Prof. S. Bilinski, D. Kurepa

B. OTHER INVITED PARTICIPANTS.

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION (UNESCO)	Prof. P. Auger, F. Le Lionnais

INTERNATIONAL COUNCIL OF SCIENTIFIC UNIONS (ICSU)	Dr. R. Fraser (in a private capacity)
INTERNATIONAL UNION FOR THEORETICAL AND APPLIED MECHANICS (IUTAM)	Prof. J. M. Burgers
INTERNATIONAL UNION FOR LOGIC, METHODOLOGY AND PHILOSOPHY OF THE SCIENCES (IUPS)	Prof. A. Heyting

Credentials of the Delegates were found in order.

Invitations to join the Union were sent well in advance of the meeting to the following countries : Brazil, Bulgaria, Chile, China, Colombia, Czechoslovakia, Egypt, Eire, Hungary, Iran, Poland, Portugal, Rumania, Turkey, Union of South Africa, U. S. S. R., Uruguay, Venezuela. The invitations were accompanied by copies of all printed documents of the Union.

2. **Action on questions concerning membership.**

The Chairman, on behalf of the Executive Committee, recommended the acceptance of the following applications for membership in the Union :

ICELAND (July 8, 1954)

Nat. Adh. Org. : The Icelandic Mathematical Society.

Nat. Comm. for Math. : Leifur Asgeirsson, Gudmundur Arnlaugusson, Gunnar Bodvarsson.

Group : one.

PORTUGAL (August 4, 1954)

Nat. Adh. Org. : Instituto de Alta Cultura.

Nat. Comm. for Math. : not indicated.

Group : one.

BRAZIL (August 13, 1954)

Nat. Adh. Org. : Sociedade Matematica de Sao Paulo.

Nat. Comm. for Math. : Joao Augusto Breves Filho (Sao Paulo), Candido L. da Silva Dias (Sao Paulo), Edison Farah (Sao Paulo), Lelio I. Gama (Rio de Janeiro), Leopoldo Nachbin, Chairman (Rio de Janeiro), Ary Nunes Tiethbol (Porto Alegre).

Group : one.

The General Assembly voted unanimously in favor of these applications; Iceland and Brazil became regular members of the Union as of September 1, 1954; Portugal will become a regular member immediately after the communication to the Secretariat of the names of the Members of the National Committee for Mathematics.

In consequence of this vote the present membership of IMU is as follows (Groups are indicated in parenthesis after each name) : Argentina (I), Australia (I), Austria (II), Belgium (III), Brazil (I), Canada (II), Cuba (I), Denmark (II), Finland (I), France (IV), Germany (IV), Great Britain (V), Greece (I), Iceland (I), India (III), Israel (II), Italy (IV), Japan (IV), Malaya-Singapore (I), Mexico (I), Netherlands (II), Norway (I), Pakistan (II), Peru (I), Portugal (I), Spain (II), Sweden (II), Switzerland (II), U. S. A. (V), Yoguslavia (II).

The position of Peru was examined and Resolutions n^0 1 (see Appendix) was adopted.

3. Appointment by the President of Working Committees.

On the suggestion of the President the appointment of the following Working Committees was approved by the General Assembly :

(*a*) *Drafting Committee*

Hille, Chairman ; Chatelet, Davenport

(*b*) *Budget Committee*

Bureau, Chairman; Bompiani, Chandrasekharan, Coxeter, Hofreiter.

4. Resolution to form Nominating Committee—Implementation of same.

Resolutions n⁰ 2 on elections in 1954, and n⁰ 3 nomination of Commissions (see Appendix) were carried.

On the President's suggestion the following names were accepted by the Assembly for the *Nominating Committee*: Stone, Chairman; Iyanaga, Leray, Norlund, Terracini.

5. Reception of Reports.

No reports (besides those already printed) were presented.

6. Action on News Bulletin.

Resolution n⁰ 4 (see Appendix) on publication of IMU news was adopted. Chandrasekharan suggested also a large dissemination of IMU news in mathematical journals.

7. Budget for 1955-1958—Discussion of contributing membership.

The Secretary explained the printed and distributed financial Report for 1952, 1953, the present financial situation of the Union (whose Emergency and Reserve Fund will be, at the end of the year, almost double what it was to be expected on the basis of the estimates of the First General Assembly); the extra-expenditures connected with the increased membership and the preparation of this Assembly; the prospective new items of expenditure (President's expenses; contribution to ICSU; expenses for chartered accounts). On this basis the Executive Committee submitted to the Assembly a tentative budget for the years 1955-58 to be examined by the Budget Committee. In the final session the General Assembly unanimously approved the distribution of IMU income for the years 1955-58 suggested by the Budget Committee (see Appendix Resolution 15).

During the discussion, Dr. Fraser hinted at the possibility of getting some financial help from UNESCO through ICSU for the preparation of a General Assembly.

President Stone observed that the margin between expenses and income in the years 1955-58 will become pretty narrow; while he did not favour increasing now the annual dues he indicated that as the Union becomes more active it will need to have a larger income from dues; and he commented that in the meantime the possibility of getting individual contributors should be explored.

8. Proposals regarding symposia.

The text of the resolution on symposia submitted to the Assembly by the Executive Committee was explained by Jessen. After an extensive discussion, in which MacLane, Auger, Stone, McShane, Cartwright, Chandrasekharan, Denjoy, Leray, Burgers, Fraser took part, the Resolution on General Rules for Colloquia (Appendix, n^0 5) was unanimously adopted.

9. Work of the Executive Committee and Commissions—addition and elimination of Commissions.

The printed Reports have already detailed the activities of the Executive Committee and of the Commissions.

On this basis, President Stone suggested discharging the Commission on Mathematical Symbols. After a discussion, in which Sansone, Stone, Denjoy, Hodge, Amira, Terracini, Severi, Kurepa took part, Resolution n^0 6 (Appendix) was unanimously adopted.

On President Stone's suggestion the reorganization and consolidation of the Commissions on Abstracting and Reviewing and on Dissemination of Mathematical Knowledge in a single Commission was examined by the Assembly and Resolution n^0 7 (Appendix) was finally adopted unanimously.

President Stone then reported on the difficulties experienced by the International Commission for Mathematical Instruction (ICMI). In order to examine carefully the recommendations of the Executive Committee to the General Assembly a working Committee, composed of Stone, Chandrasekharan, Chatelet, Kurepa, Sansone, was appointed to report later to the Assembly. The Com-

mittee's suggestions were finally approved unanimously as in Resolution n^0 8 (Appendix).

On a suggestion by Miss Cartwright the Assembly agreed to reverse the order of discussion of n^0 10 and n^0 11 of the Agenda.

10. Topics suggested by the National Adhering Organizations.

Kurepa discussed his proposal on the "Production of a film about the role of Mathematics and Mathematicians at the present time". Signorini, Chatelet, Newman, Burgers, Stone raised questions about the usefulness, the feasibility and the cost of such a project. The General Assembly finally adopted Resolution n^0 9 (Appendix).

Terracini asked the President for an official interpretation of the Statutes concerning the meaning of the term "General Assembly" when the delegates are not physically assembled. President Stone stated that practical considerations favored the working interpretation that the National Committees are deemed to constitute the General Assembly when the latter is not in session; he proposed that if there were no objection this working interpretation should remain effective until modified by an amendment to the Statutes. No objection was voiced.

11. Discussion of relations with ICSU and other Unions.

On the problem of the reorganization of the Unions into ICSU (von Muralt plan) the Assembly adopted unanimously Resolution n^0 10 (Appendix).

As a consequence of some remarks by Dr. Fraser (ICSU) on the work of the Commissions, Resolution n^0 11 (Appendix) was adopted.

Following a suggestion of President Stone, adopted in substance by the Assembly, the representatives of IMU to ICSU should be, as a rule, the President and the Secretary, with the power in the President to indicate other persons if necessary.

On the question of relations with other Unions, Burgers (IUTAM) stressed the importance of these relations : subjects of common interest to IMU and IUTAM (and other Unions, like IUPAP) could be the definition of symbols, the study of mathematical problems in calculating machines, non-linear differential equations ; joint symposia on these subjects could be held. As a consequence Resolution n⁰ 12 (Appendix) on a joint-Commission of IMU and IUTAM was unanimously adopted by the Assembly.

The situation of the International Union for Methodology and Philosophy of the Sciences (IUPS) was presented by Heyting. As IUPS has not been admitted to ICSU as an independent Union, it has been suggested by ICSU authorities that an agreement be made with the IMU so as to make it possible to have direct subventions from ICSU. President Stone expressed to Prof. Heyting the assurance that the IMU expects to give continuing attention to the problem of establishing suitable cooperative arrangements with IUPS as well as other Unions and other organizations which have interests in common with IMU.

Jessen suggested that to make relations with other Unions more likely, attention of the National Adhering Organizations to IMU should be called to the fact that the National Committees for Mathematics should be representative of all branches of mathematics.

Severi reported on the present situation of the International Computing Center ; Auger added information about the nations which have signed or ratified the convention establishing the Center ; Papaeoannu announced that Greece will soon ratify the convention. The Assembly adopted Resolution n⁰ 13 (Appendix).

At this point in the meeting Kloostermann expressed the thanks of the Organizing Committee of the International Congress in Amsterdam for the help received from UNESCO, ICSU and the Executive Committee of IMU ; and suggested the appointment of a joint-Committee to determine the time and place of the next International Congress.

Resolution n° 14 (Appendix) was unanimously carried.

12. Elections.

(1) President Stone reported to the Assembly the findings of the Nominating Committee. The following names for the Officers and elected Members of the Executive Committee were offered for the period January, 1, 1955—December, 31, 1958.

President : H. Hopf ; First Vice-President : A. Denjoy ; Second Vice-President : W. V. D. Hodge ; Secretary : E. Bompiani ; Elected Members : K. Chandrasekharan, J. F. Koksma, S. Mac Lane.

No other names were suggested to the Nominating Committee.

The General Assembly unanimously elected the slate proposed by the Nominating Committee.

(2) The representatives of IMU to ICSU will be the President and the Secretary ; the Executive Committee may appoint substitutes for these representatives when necessary.

(3) The elections of the various Commissions for the period January 1, 1955—December 31, 1958 gave the following unanimous results :

Commission on Mathematical Publications :

E. Hille, Chairman ; B. A. Amira, W. Fenchel, members.

Commission on the Exchange of Mathematicians :

H. Davenport, Chairman ; S. Iyanaga, J. R. Kline, members.

Commission for the World Directory of Mathematicians :

M. H. Stone, Chairman ; P. Belgodere, E. Bompiani, W. V. D. Hodge, members.

International Commission for Mathematical Instruction :

H. Behnke, Chairman ; Y. Akizuki, G. Ascoli, P. J. Dubreil, J. C. H. Gerretsen, R. L. Jeffery, D. Kurepe, E. A. Maxwell, Ram Behari, M. H. Stone, members-at-large.

The President recalled that in accordance with the Statutes each Commission may co-opt additional members.

(4) At this point of the meeting and before leaving the chair, Hodge took the opportunity to thank President Stone for what he did for the Union both before its inception and during these three years of existence. Resuming the chair, President Stone addressed appropriate words of thanks to Professor Hodge, the Members of Executive Committee and the Assembly for their co-operation. Dr. Fraser expressed his gratitude for having been invited to this Assembly and his satisfaction for the work done in this first interval of existence of the Union with its President and Secretary.

13. Resolutions approving Reports.

The Assembly approved the final draft of the resolutions prepared by the Working Committees, Resolution n⁰ 15 (Appendix) on the Budget for the period 1955-58 and Resolution n⁰ 16 (Appendix) approving the printed Reports submitted to the Assembly by the Executive Committee.

14. Resolution on Congress—Next meeting of General Assembly.

The Assembly authorized the Executive Committee, by Resolution n⁰ 17, to fix the time and place of the third General Assembly in connection with the next International Congress of Mathematicians.

15. Courtesy resolutions.

The Assembly approved Resolution n⁰ 18 (Appendix) thanking Professors Kloostermann, Koksma, Loonstra, Zaanen for their co-operation in the preparation of the present Assembly; the Burgomaster and the Wethouders of the Hague, and the President, Curator, and Rector Magnificus of the Technical University of Delft for the receptions offered to the Delegates.

In addition the General Assembly sent telegrams of appreciation of their services and of good wishes to Professors Emile Borel and Erich Kamke, Vice-Presidents of the IMU, who could not attend the Assembly because of their physical condition (see Appendix).

APPENDIX

RESOLUTIONS OF THE SECOND GENERAL ASSEMBLY OF THE INTERNATIONAL MATHEMATICAL UNION THE HAGUE

August 31—*September* 1, 1954

1. Resolution on Peru.

Resolved :

That in accordance with article 22 of the Statutes the name of Peru be removed from the list of voting members of the IMU.

2. Resolution on elections in 1954.

Resolved :

That nominations for the period 1955-1958 for :

(1) the Officers of the Union,

(2) the additional members of the Executive Committee,

(3) the representatives of IMU to ICSU,

be prepared and submitted for action to this Assembly by a Nominating Committee, consisting of the President ex-officio and four additional members elected by this Assembly on nomination of the Executive Committee. Proposals for the list of nominations may be made to the Nominating Committee by any member of the General Assembly. No name which has not been so proposed may later be put in nomination for the posts to be filled at these elections. Every name which has been so proposed should be reported by the Nominating Committee to the General Assembly.

3. Resolution on nomination of Commissions.

Resolved :

That nominations for the Members and Chairmen of the various Commissions of IMU be submitted by the President for election by the General Assembly.

4. Resolution on publication of IMU News.

Resolved :

That there be appropriated annually over the period 1955-1958 a sum not exceeding $300 to be expended by the Secretary, on the advice of the Executive Committee, for the purpose of publishing and disseminating news of the activities of IMU and that this sum be accounted for separately.

5. Resolution on General rules for Colloquia.

Resolved :

That the General Assembly adopt the following rules for organization of colloquia :

1. A colloquium is thought of as a meeting of a limited number of invited participants who are either experts or promising younger scientists, working in a field of current mathematical activity. This definition does not preclude the presence of a small number of other interested persons.

2. The Executive Committee of the Union may take the initiative in the organization of a colloquium ; but in general the Union will act as co-sponsor for colloquia suggested by the National Adhering Organizations or by interested groups of mathematicians. The Executive Committee decides which colloquia should be organized or supported by the Union and seeks to obtain the necessary funds. The aim will be to support at least one colloquium each year. From time to time the Union may participate in the organization of joint colloquia with other Unions.

3. Organizations or groups of mathematicians who desire the participation of the Union in the organization of a colloquium should send their proposals to the Secretary of the Union. The proposal must state the subject, the place and the intended date of the colloquium, and must include a preliminary budget indicating the support that may be expected from other sources and the support from the Union that will be needed for the successful organization of the colloquium. The support from the Union must be

used exclusively for travelling and maintenance expenses of invited participants.

4. The Executive Committee will discuss the future program of colloquia each year on the basis of proposals received before February 1 (in 1954 before November 1). In setting up the program the Executive Committee will take into account the interest of the various subjects from the point of view of the actual state of science, the scientific circumstances in the countries in which the colloquium might take place, and what colloquia have taken place in recent years. The Executive Committee will aim at a rotation of aid among the principal geographical areas of mathematical activity, and among the various branches of mathematics. The Executive Committee will aim at generous support of the colloquia of which the Union acts as a co-sponsor.

5. When the Executive Committee takes the initiative for a colloquium it appoints an organizing Committee including at least one of its own members.

6. When the Union acts as a co-sponsor of a colloquium proposed by a National Adhering Organization or a local group, the Executive Committee will appoint one of its members and one or two other mathematicians to join the national or local Organizing Committee for the colloquium. These members are to participate in the scientific arrangements, and in particular in the selection of the participants for the meeting.

7. Information about colloquia to be organized or supported by the Union shall be suitably published and sent to the National Committees for Mathematics, which may call the attention of the organizing Committee to experts whose participation is considered desirable.

6. Resolution on discharging the Commission on Mathematical Symbols.

Resolved :

That the Commission on a Directory of Mathematical Symbols be discharged.

7. Resolution on forming a new Commission on Scientific Publications.

Resolved :

That the Commissions on Dissemination of Mathematical Knowledge and on Abstracting and Reviewing be consolidated into a single Commission on Scientific Publications, with the following terms of reference :

(*a*) The Commission shall advise the Executive Committee and the General Assembly on all questions concerning scientific publication which may come before them ;

(*b*) The Commission shall study the desirability of a program of scientific publication under the auspices of IMU and shall report its conclusions to the Executive Committee and the General Assembly not later than December 31, 1956 ;

(*c*) The Commission shall study problems of scientific publication in the field of Mathematics with a view to determining in what way, if any, IMU can contribute to their solution ;

(*d*) The Commission shall be authorized to arrange, with the approval of the Executive Committee, meetings of editors and other interested persons for the purpose of discussing specific problems of scientific publication in the field of Mathematics.

(*e*) The Commission shall prepare a report reviewing the use of the principal mathematical symbols. It may establish contact with any international organization concerned with the use of mathematical symbols from the point of view of technical applications.

8. Resolution on the Commission for Mathematical Instruction.

Resolved :

That the terms of reference and the constitution of the International Commission for Mathematical Instruction be clarified and determined as follows :

(a) The Commission shall consist of ten members-at-large elected by the General Assembly on nomination of the President of IMU, and of two national delegates named by each of the National Committees for Mathematics as specified below;

(b) The officers of the Commission shall consist of a President, a Secretary and two Vice-Presidents. The President shall be elected by the General Assembly on the nomination of the President of the IMU from the membership at large of the Commission;

(c) The Executive Committee of the Commission shall consist of the Officers of the Commission together with three additional Members elected by the membership of the Commission;

(d) During his life-time Professor Fehr (Geneva) shall remain Honorary President of the Commission in recognition of his long-time interest in the cause of Mathematical education and his devoted services to it;

(e) In all other respects the Commission shall make its own decisions as to its internal organization and rules of procedure;

(f) Any National Adhering Organization wishing to support or encourage the work of the Commission may create, in agreement with its National Committee, a subcommittee to maintain liaison with the Commission in all matters pertinent to its affairs. The National Adhering Organization in question shall designate two members of the said subcommittee, if created, to serve as delegated members of the Commission;

(g) The Commission shall be charged with the conduct of the activities of IMU bearing on mathematical and scientific education, and shall take the initiative in inaugurating appropriate programs designed to further the sound development of mathematical education at all levels and to secure public appreciation of its importance;

(h) The budget of the Commission shall be submitted to the Executive Committee of IMU and the General Assembly for approval at such times as may be determined by agreement between the Commission and the Executive Committee;

(i) The Commission shall file an annual report of its activities with the Executive Committee of IMU and shall file a quadrennial report at each regular meeting of the General Assembly.

9. Resolution on the production of a film about the role of Mathematics and Mathematicians at the present time.

Resolved :

That the proposal for the production of a film concerning the importance of mathematics be referred for further study to the Commission for Mathematical Instruction.

10. Resolution on relations with other Unions.

Resolved :

That the General Assembly report to ICSU its view that the reorganization proposed by Professor von Muralt should not be undertaken without further study, designed inter alia to provide, in a more realistic and farsighted way, for the representation and co-ordination of the increasingly numerous and diverse mathematical activities in all branches of science.

11. Resolution on activities of Commissions.

Resolved :

That every Commission, as constituted at the close of the present General Assembly for the period 1955-58, be required to submit to the Executive Committee in time for its next meeting a detailed programme and a detailed budget for its activities during the year 1956.

12. Resolution on a Joint Commission of IMU and IUTAM.

This Assembly requests the Executive Committee to explore, in consultation with IUTAM, the desirability of making a proposal to ICSU to appoint a joint-Commission of the IMU and the IUTAM for carrying on specific activities of common interest ; and to take appropriate action thereon.

13. Resolution on the International Computing Center.

Resolved :

That the Executive Committee be requested to consider the relations between IMU and the International Computing Center when it comes into legal existence, and to establish such relations as seem best to it.

14. Resolution on Joint Committee for time and place of next Congress.

Resolved :

That this Assembly delegate the President and the Secretary of the IMU and a third representative in the person of Professor Iyanaga to constitute together with the President and the Secretary of the International Congress of Mathematicians 1954 a Joint-Committee of the Union and the 1954 Congress with the duty of considering the time and place of the next International Congress of Mathematicians and reporting its recommendations thereon to the Plenary Session of the 1954 Congress on Thursday, September 9, 1954.

15. Resolution on the budget for 1955-58.

(1) The second General Assembly authorizes the following annual expenditure for 1955-58 against IMU funds :

		$
(I)	Secretarial help	1.800
(II)	Office expenses	600
(III)	Travelling expenses	750
(IV)	President's expenses	250
(V)	ICSU contribution (approximately)	50
(VI)	Publication and dissemination of IMU news	300
(VII)	Accountant's fee	150
		3.900
(VIII)	Preparation of the General Assembly in 1958	350

(2) The Assembly authorizes the Executive Committee to reallocate any sums not used for the purpose designed, for use for one of the other purposes in the same year or in a later year during the period 1955-58.

(3) The Assembly resolves that the sum of $ 500 allocated for office expenses for the current year be increased to $ 1.000 in order to meet expenses incurred in the preparations for the present Assembly.

16. Approval of the printed reports.

This Assembly approves all the published reports of the Committes and Commissions of the IMU presented to this Assembly, with the modifications, if any, decided upon by the Assembly.

17. Motion of next General Assembly.

This Assembly authorizes the Executive Committee to call the third General Assembly of the IMU at the same time and place as those decided upon for the meetings of the next International Congress of Mathematicians by the joint Committee of the IMU and the Congress already appointed by the Assembly.

18. Courtesy resolutions.

Resolved :

That the Second General Assembly of the International Union presents its hearty thanks to its hosts. In particular the Assembly wishes to express its gratitude to the following persons and organizations.

To Professors Kloostermann, Koksma, Loonstra and Zaanen for the excellent local arrangements.

To the Burgomaster and the Wethouders of the Hague for the gracious reception in the Town Hall.

To the Rector of the Technical University of Delft for his cordial hospitality.